Chapter 1

The Pythagorean Theorem

Euclid I.47 and VI.31

What are the chances that Pythagoras, or even Thales before him, *knew* the famous mathematical proposition that has come down to us under the name "Pythagorean theorem?" The question is not whether he *proved* it, or even if he stated it in the form of a theorem. Could either of these men who lived in the sixth century BCE have *understood* it? And if so, in what senses? Since, until recently, scholarship has tended to discredit "Pythagoras" not only with regard to the theorem but to any contributions to mathematics, could a circumstantial case be constructed that points to such understanding, contrary to the prevailing views? To even approach the question, we must be very clear about what we are looking for. In these earliest chapters of the Greeks in geometry, perhaps the mathematical proposition was grasped differently than in the forms in which it has come down to us in our modern education; indeed, it may have been grasped differently in Euclid himself.

So, let us begin by exploring the Pythagorean theorem. How did Euclid present it at the end of the fourth century BCE? How might it have been understood two-and-a-half centuries earlier by the philosophers of the sixth century BCE? Or to put the matter differently, *what do you know when you know the Pythagorean theorem?*

First, let us be clear that the theorem we are exploring is the one that claims a law-like relation among the sides of a right triangle, a law-like relation between line lengths and figures drawn on the sides of a right triangle. It has been my experience that most educated people today respond to this question about the Pythagorean theorem by producing a formula—$\mathbf{a^2 + b^2 = c^2}$—and offer as an explanation that when asked to calculate the length of a side of a right triangle given the lengths of two other sides, they apply the formula to derive the answer. But this formula is algebra; the Greeks of the time of Euclid, and certainly before, did not have algebra, nor does the surviving evidence show that they thought in algebraic terms.[1] So this poses a false start from the outset. If we begin with Euclid, and then hope to look back into the earlier investigations, we must focus on the relation between line lengths and the areas of figures constructed from them. Can Pythagoras, or Thales, be connected with an understanding that brings together the lengths of the lines with the areas of figures constructed on them, and if so, how and why?

To begin this exploration, I shall set out (A) the most famous presentation of the theorem from Euclid's Book I, Proposition 47, and then explain the claims that must be connected to effect the proof. I next set out the mathematical intuitions[2] that anyone would be assumed to know to connect what is connected in *this* proof. Next I shall set out (B) the largely neglected, second version of the proof, sometimes called the "enlargement" of the Pythagorean theorem as it is presented in Euclid VI.31, and follow this by explaining the claims that must be connected to effect this proof; I then set out the mathematical intuitions that anyone would be assumed to have to know to connect what is connected in the proof. Next (C), I explain the related concepts of "ratio," "proportions," and "mean proportional" as they apply to this discussion. Afterward (D), I try to clarify further the idea of the mean proportional (or geometric mean) by contrasting it with an arithmetical mean. And finally (E), I provide an overview and summary connecting to the idea of the *metaphysics* of the hypotenuse theorem that anticipate the arguments of chapters 2 and 3, which offer a broader and wider picture of how Pythagoras, and Thales before him, plausibly knew the theorem. Taking this approach puts us in a position to know precisely what we are looking for, had Thales or Pythagoras grasped these relations between line lengths and the areas of figures constructed on them.

A

Euclid: The Pythagorean Theorem I.47

First, (i) I will present the proof of the theorem as it appears in Euclid I.47, then (ii) I will explain the strategy of the proof in a reflective way, and third (iii) I will set out the *ideas* that must be connected—the *intuitions*—whose connections are the proof. The importance of this third approach is central to my project because without being misled by the ambiguous question "Could a sixth century Greek *prove* this theorem?"—appealing to Euclid as a paradigm of "proof"—we are asking instead "Could a sixth century Greek have grasped the relevant ideas and connected them?" And if so, what is the nature of the evidence for them, and moreover, what does this tell us about *reasoning*, formal and informal, and why the Greeks might have been investigating these things?

(i) The Pythagorean theorem of Euclid I.47 (following Heath):

> Ἐν τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν τετραγώνοις.
>
> "In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle."

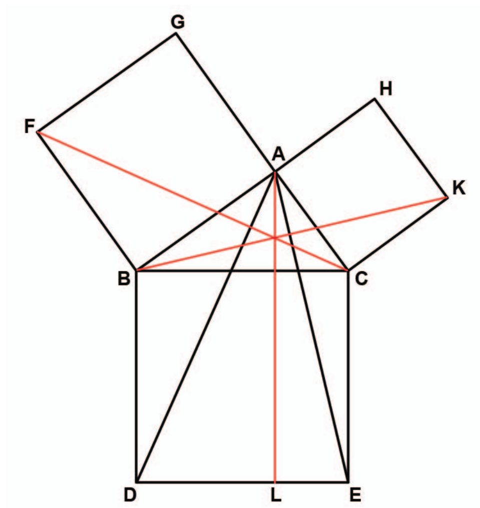

Figure 1.1.

Let ABC be a right-angled triangle having the angle BAC right;

I say that the square on BC is equal to the squares on BA, AC.

For let there be described on BC the square BDEC, and on BA, AC the squares GB, HC; **[I.46]** through A let AL be drawn parallel to either BD or CE, and let AD, FC be joined.

Then, since each of the angles BAC, BAG is right, it follows that with a straight line BA, and at the point A on it, the two straight lines AC, AG not lying on the same side make the adjacent angles equal to two right angles;

Therefore CA is in a straight line with AG. **[I.14]**

For the same reason BA is also in a straight line with AH. And, since the angle DBC is equal to the angle FBA, for each right, let the angle ABC be added;

Therefore, the whole angle DBA is equal to the whole angle FBC. **[C.N. 2]**

And since DB is equal to BC, and FB to BA, the two sides AB, BD are equal to the two sides FB, BC respectively;

and the angle ABD is equal to the right angle FBC;

Therefore, the base AD is equal to the base FC, and the triangle ABD is equal to the triangle FBC. **[I.4]**

Now the parallelogram BL is double the triangle ABD, for they have the same base BD and are in the same parallels BD, AL. **[I.41]**

And the square GB is double of the triangle FBC, for they again have the same base FB, and are in the same parallels FB, GC. **[I.41]**

[But the doubles of equals are equal to one another.]

Therefore, the parallelogram BL is also equal to the square GB.

Similarly, if AE, BK be joined, the parallelogram CL can also be proved equal to the square HC; therefore, the whole square BDEC is equal to two squares GB, HC. **[C.N. 2]**

And the square BDEC is described on BC, and the squares GB, HC on BA, AC.

Therefore the square on the side BC is equal to the squares on the sides BA, AC.

Therefore etc. **QED [OED]**

(ii) Reflections on the strategies of Euclid I.47:

The strategy of the proof is to create, first, squares on each side of a right triangle. Next, draw a perpendicular line from the right angle of the triangle to the base of the square on the hypotenuse BC. The result is that the square divides into two rectangles, each of which will be shown to be equal to the square drawn on each of the remaining two sides, respectively, of the original triangle ABC.

To show this, then, a line is drawn from the right angle, vertex A, to point L parallel to both sides of the square drawn on the hypotenuse, BD and CE. This creates rectangle BL, which is then divided by drawing a line from point A to D, the left corner of the square, creating at the same time triangle ABD. The next step is to show that rectangle BL is twice the area of triangle ABD because both share the same base BD between the same two parallel lines. Even now we must keep in mind that the strategy of the proof is to show that the square on the hypotenuse BC, now divided into two rectangles, is equal to the sum of the squares on the legs AB and AC.

Next, the argument shows that triangle ABD is equal to triangle FBC, because sides AB and BD are together equal to sides FB and BC, the angle ABD is equal to the angle FBC, and base AD is equal to base FC. This is side-angle-side (SAS) equality, which Euclid proves in theorem I.4. And because triangle FBC and square GB share the same base FB between the same parallels FB, GC, the square is double the area of triangle FBC. And then, since triangles FBC and ABD are equal, the rectangle BL that is also double the triangle ABD is also equal to the square GB, because things equal to the same thing are also equal to each other. At this point, the proof has shown that the square on the longer side of the right triangle is equal to the larger rectangle into which the square on the hypotenuse is divided.

Then, AE and BK are joined, creating triangles AEC and BKC, and they are shown to be equal to each other, ceteris paribus. And because both rectangle CL and triangle AEC share the same base CE, and are between the same parallel lines, the rectangle is double the area of triangle AEC. And since the triangle AEC is equal to the triangle BKC, the square HC must be double the area of triangle BKC, since both share the same base KC between the same parallel lines, and thus the rectangle CL is equal to the square HC, since things equal to the same thing are also equal to each other.

Finally, the proof ties together these two parts by showing that since the rectangle BL is equal to the square GB, and the rectangle CL is equal to the square HC, and the two rectangles BL and CL placed together comprise the largest square BDEC, thus the squares on each of the two sides, taken together, are equal to the square on the hypotenuse.

What follows, immediately below, then, is the sequence of this proof diagrams:

These two triangles are of equal area because they share two sides equal and the angle between those sides (SAS equality, I.4):

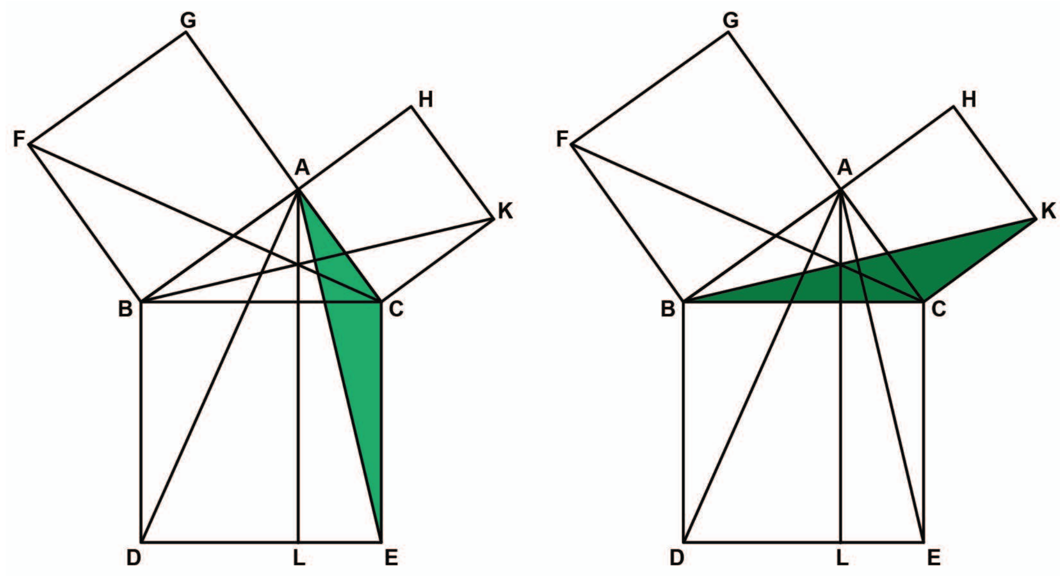

Figure 1.2.

The rectangle has twice the area of the triangle since they share the same base and are within the same parallel lines AL and CE (I.41).

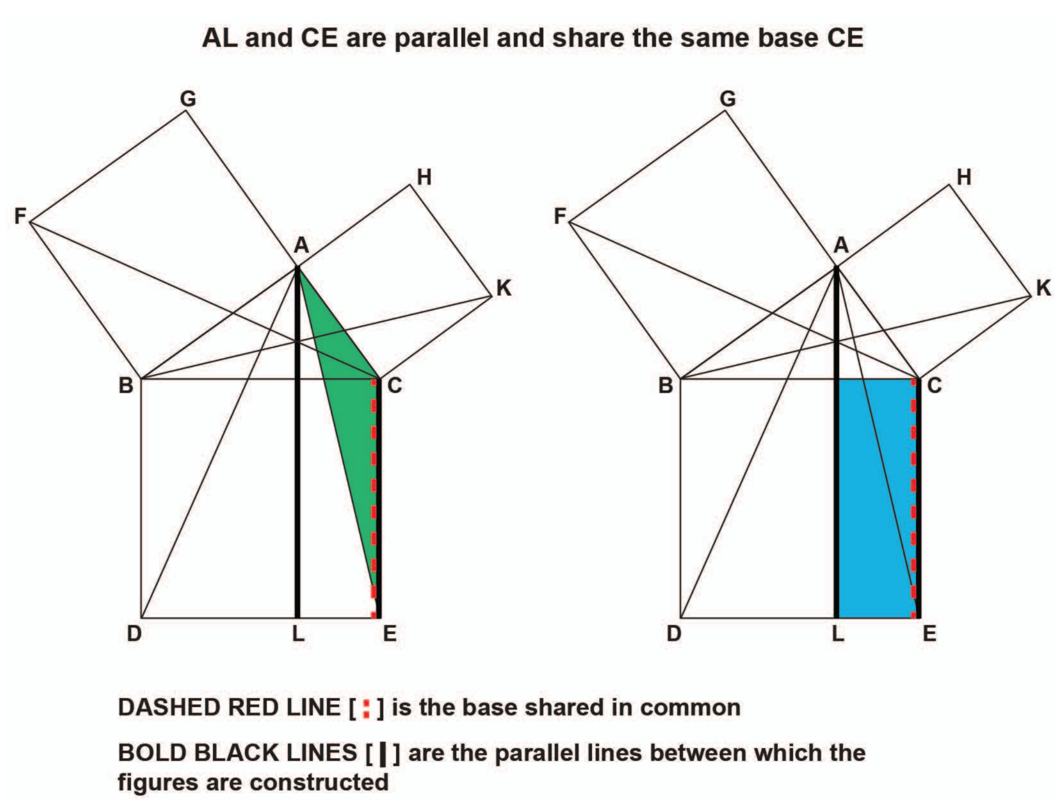

Figure 1.3.

The square has twice the area of the triangle, since they too share the same base and are within the same parallel lines KC and HB:

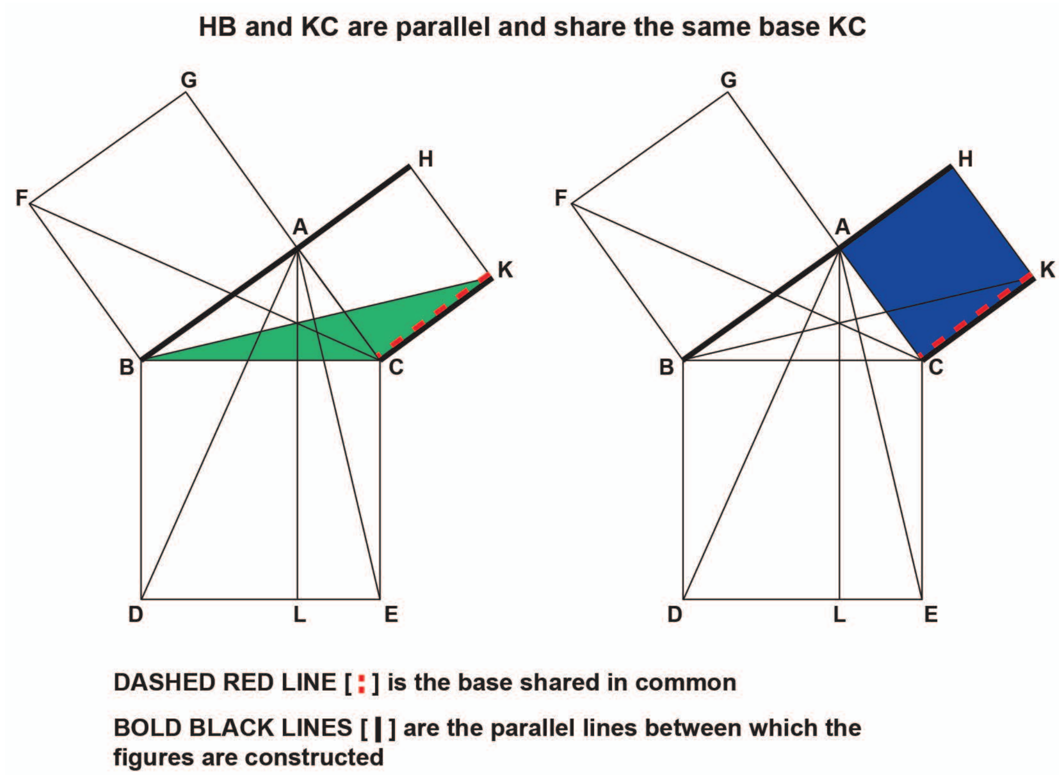

Figure 1.4.

Thus, the square on the shortest side of the right triangle has the same area of the smaller rectangle into which the square on the hypotenuse has been divided because each is double the area of the triangle that shares the same base, between the same parallel lines, and each of those triangles is equal to one another. The rectangle and square are equal because they are double triangles equal to one another.

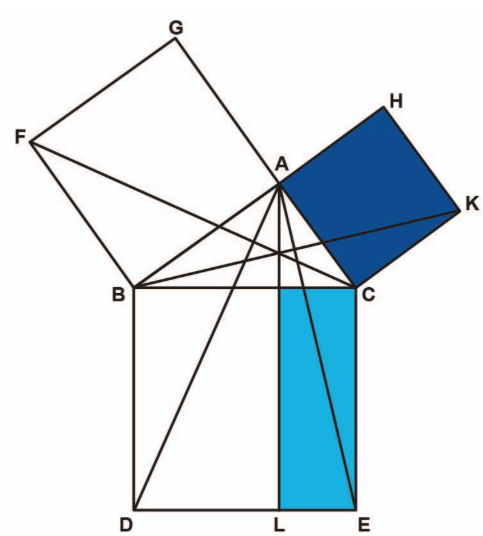

Figure 1.5.

At this point I wish to modify what I have just written. I have expressed this point of equality of figures as equalities of area, though this is not Euclid's way of expressing the matter. He customarily speaks of *figures* being equal, not areas. Looking back to the sequence of propositions beginning with I.35, this is a point that needs to be considered. When he says that parallelograms are equal or triangles are equal, it is the figure that is equal and not some number attached to it. By appeal to Common-Notion 4, things that coincide with one another are equal to one another. From there, however, the remaining Common-Notions allow Euclid to conclude that figures that do not coincide are also equal. This point is worth emphasizing for we are looking at figures, square AK and rectangle CL, that are equal despite their very different shape.

Now the same strategy proceeds to show that the square on the longer side of the right triangle is equal in area to the larger rectangle into which the square on the hypotenuse is divided.

First, these two triangles are shown to be equal in area because they too share two sides in common and the angle between them (SAS equality, I.4):

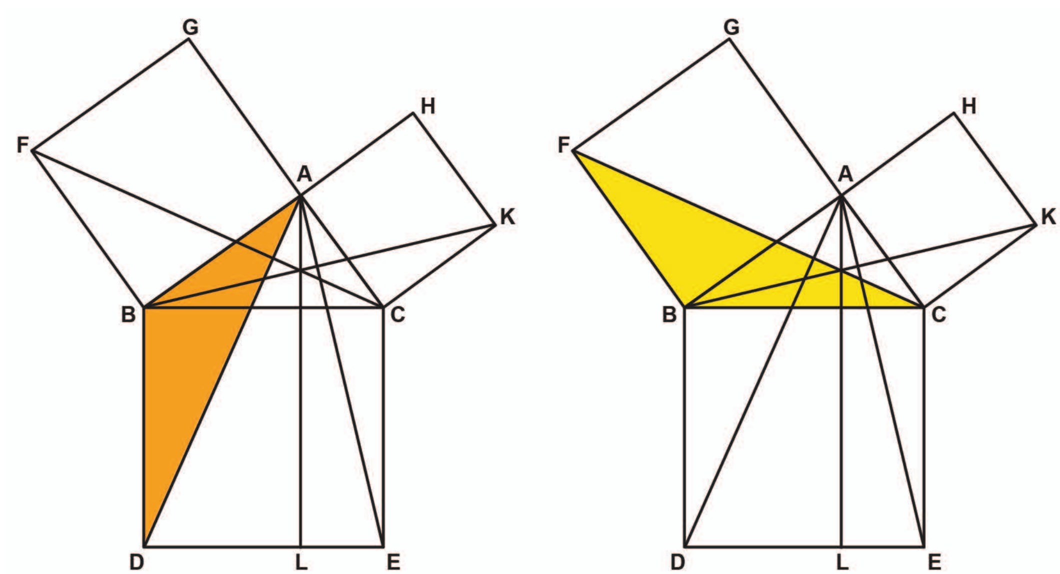

Figure 1.6.

Since both the triangle and the rectangle share the same base and are between the same parallel lines BC and AL, the rectangle has twice the area of the triangle:

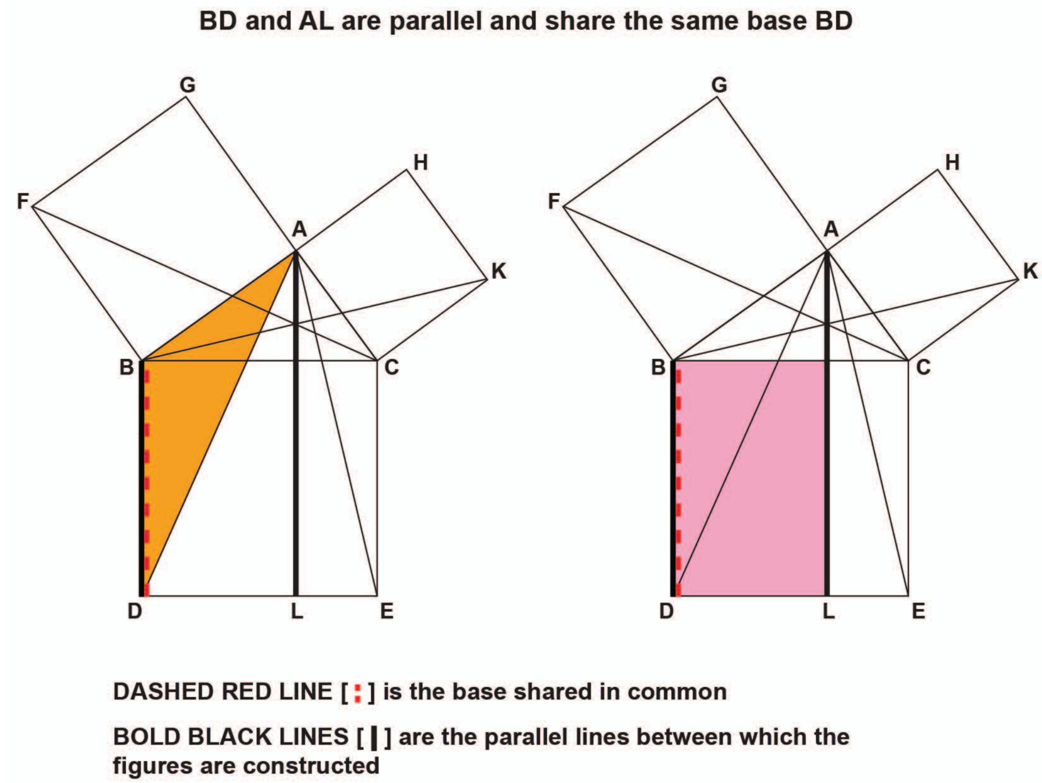

Figure 1.7.

And since both this triangle and this square share the same base and are between the same parallel lines FB and GA, the square has twice the area of the triangle:

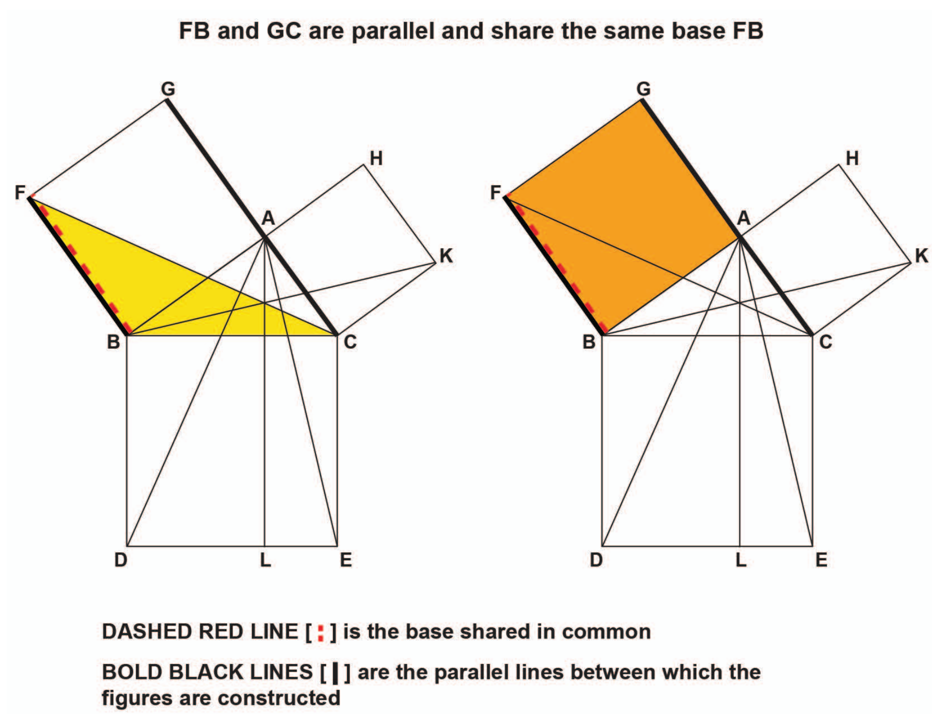

Figure 1.8.

And so the square on the longer side of the right triangle has the same area as the larger rectangle into which the square on the hypotenuse is divided, because each is double the area of the triangle that shares the same base between the same parallel lines, and each of those triangles is equal to one another:

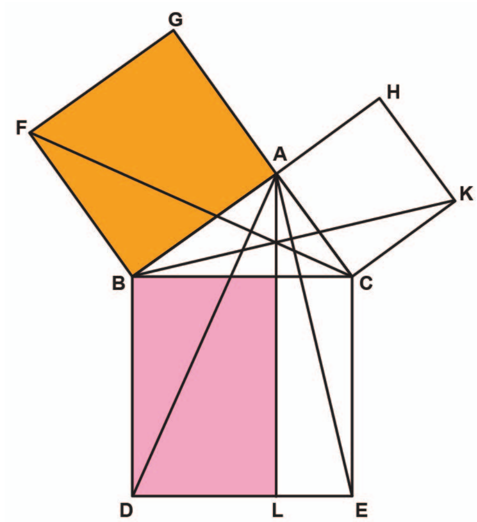

Figure 1.9.

And thus, the areas of the squares on each side of the right triangle are equal to the two rectangles, respectively, into which the square on the hypotenuse is divided:

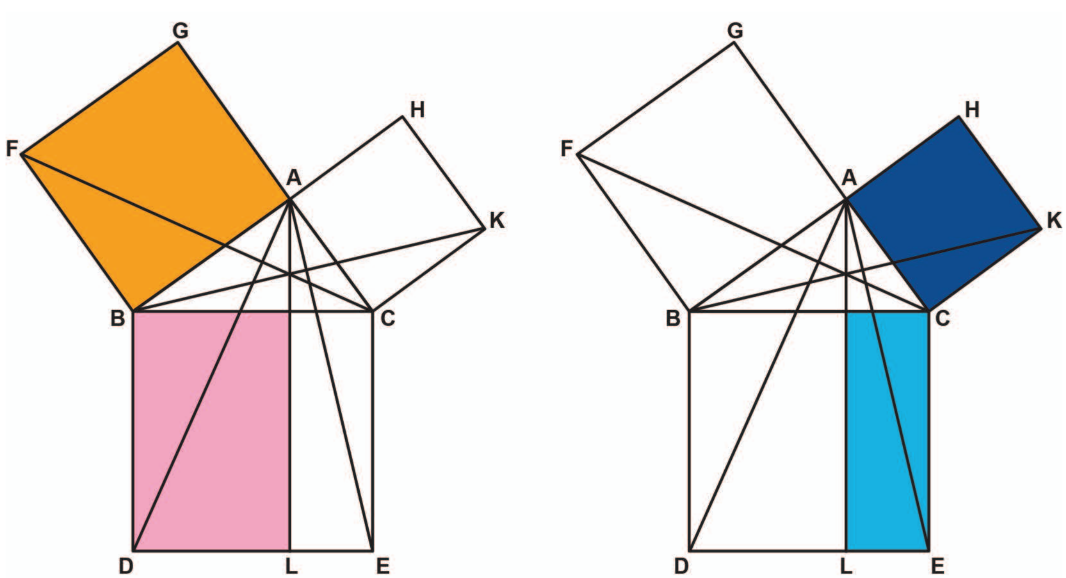

Figure 1.10.

And thus, the square on the hypotenuse is equal to the sum of the squares on the two sides:

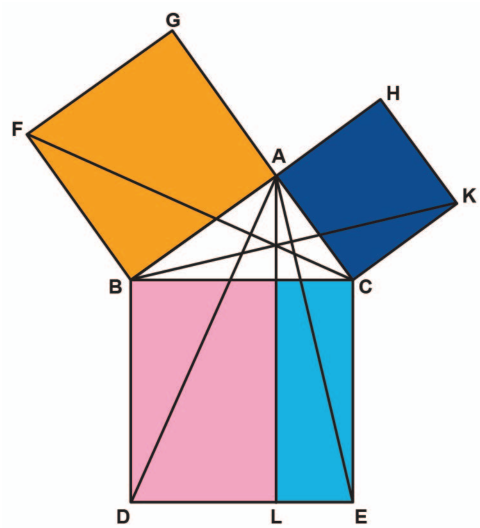

Figure 1.11.

(iii) The geometrical intuitions: the sequence of ideas that are connected in the proof:

What I have presented, immediately above, then, is the argument sequence, of the formal proof for Euclid's I.47, the so-called "Pythagorean theorem," following Heath's version of Heiberg's text. But let us now review the proof trying to get clear about the basic ideas that have to be connected to produce the sequence of thoughts that comprise *this* proof. Stated in another way, we remind ourselves that each step in a formal proof is justified by an appeal to an abstract rule—this is key to understanding the *deductive method*. These rules are appealed to in each step of Euclid's formal proofs. So we now reflect on what Euclid thinks we have to know to follow the sequence of I.47. Could it be that our old friends of the sixth century BCE grasped the ideas and their connections, whether or not they could produce such a "proof"—that is, technically speaking, this irrefragable chain of connected thoughts justified by explicitly stated rules of derivation? This is the question to be kept in mind as we continue. So, what ideas would have to be grasped?

It is a fair generalization to say about Euclid's *Elements* that the central ideas that run throughout, but especially in Books I–VI, are equality and similarity. If we may be allowed to place the themes in more modern terms, we might describe the contents of Books I–VI as being concerned with *congruence*, *areal equivalence*, and *similarity*—though there is no term in ancient Greek that corresponds with what we call "congruence." The theorem at I.47 relies on equalities and areal equivalences; the theorem at VI.31 relies on extending these ideas to include similarity, and the ratios and proportional relations that this entails, explored in Book V. The matter of *equality* among triangles is taken up in Book I propositions 4, 8, and 26. What these propositions show is that *if certain things are equal in two triangles, other things will be equal as well*. And the overall, general strategy of Book I shows that the common notions are axioms for equality and include a fundamental test, namely, that things that can be superimposed are equal; I.4 begins an exploration of equal figures that are equal via superimposition, that is, they are equal and identical in shape; I.35 shows that figures can be equal that cannot be superimposed, that is, figures can be equal but not identical in shape; I.47 shows that it is possible to have two figures of a given shape, a square, that can be equal to a third figure of the same shape.

I.4: (SAS—Side-Angle-Side) Two triangles are equal if they share two side lengths in common and the angle between them:

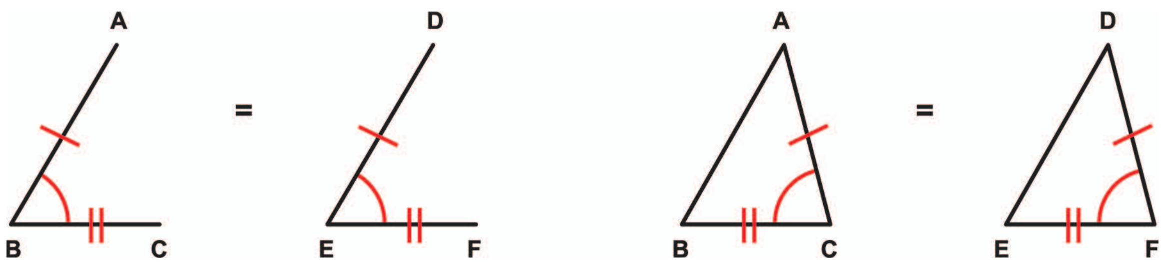

Figure 1.12.

I.8: (SSS—Side-Side-Side) Two triangles are equal (i.e., congruent) if they share the lengths of all three sides:

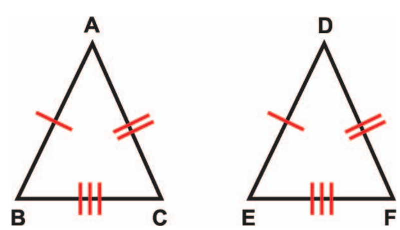

Figure 1.13.

I.26: (ASA—Angle-Side-Angle) Two triangles are equal (i.e., congruent) if they share two angles in common and the side length between them:

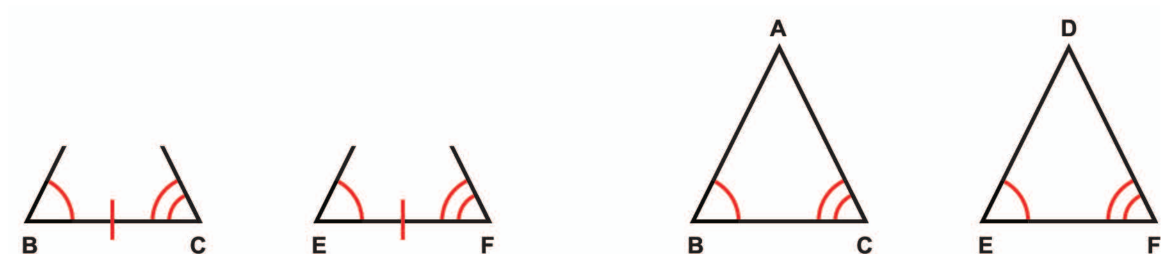

Figure 1.14.

The third theorem of equality, I.26, is explicitly credited to Thales by Proclus on the authority of Eudemus.[3] Two triangles are shown to be equal—congruent—if two sides and the angle contained by them are equal (SAS), if all three sides are equal in length (SSS), and if two angles and the side shared by them either adjoining or subtending are equal (ASA). The last of these is credited to Thales because Eudemus *inferred* it was needed for the measurement of the distance of a ship at sea. We shall investigate such a measurement in chapter 2. An understanding of it was needed to grasp the measurement, by one approach, while an understanding of similarity was needed by all the other approaches. But let us get clear about the general matter; one does not come to grasp angle-side-angle equality without having recognized side-side-side and side-angle-side equality. The equality and similarity of triangles are principles abundantly clear already by the measurement of pyramid height, as we shall also explore in the next chapter, despite the endless controversies in the scholarly literature of whether Thales or someone in the sixth century could have grasped the idea of equality or similarity.

In addition, almost certainly, the earliest attempts to prove equality—equality between triangles, for example—were by superposition; they were ἐφαρμόζειν proofs. This means that one triangle was quite literally placed on top of another, showing that the lengths *coincided* (ἐφαρμόζειν) and that the angles met at the same places and to the same degree. In the language of I.4, with the triangle ABC fitted on DEZ with point A placed on point D while straight line AB is fitted on DE, point B fits on E because AB is equal to DE. To show these kinds of equality, to quite literally make them visible (i.e., "to prove"—δείκνύμι), triangles, squares, rectangles, and all rectilinear figures were capable of being *rotated* to fit. Euclid does not like this, and tries to avoid it whenever possible—because all physical and material demonstrations are subject to imperfections—but certainly this is how "proof" began for the Greeks in the sixth century BCE and before.[4] "Proof" was a way of making something *visible*,[5] something for all to see and confirm. We must keep this in mind as we continue; the archaic Greek minds of persons such as Thales and his compatriots could envision equality by *rotating* one object—or a figure in a diagram—one over another to see equality of fit. To make this point clear, I refer to the famous "tile standard" in the southwest corner of the Athenian Agora. If one had purchased, say, a roof tile and had any doubts about whether it was standard fare, he had only to take the tile to the standard, rotate it about so it would line up with it, edge to edge and angle to angle, and test it for fit. If by visible inspection, the tile "fit," there was the *proof* of its equality. Clearly, "proof" consisted in making the equality "visible."

Figure 1.15.

Now, when a parallelogram, say a rectangle, is claimed to be double a triangle—and this means double the *area* (χωρίον)—such a demonstration cannot be effected as easily by the ἐφαρμόζειν technique. Let us keep in mind that for Euclid a χωρίον is not a number but the space contained within a figure, an area set off by a fence, as it were. Now before we follow Euclid further, let's keep in mind the question of whether and how areal equivalence could be shown by "superposition."

Let us recall how equality between figures is demonstrated in Euclid Book I, since the proof of the Pythagorean theorem at I.47 requires that two triangles be shown to be equal; then the argument is divided into two parts. First, one of the triangles is shown to be half the area of a rectangle (i.e., a parallelogram) since they are constructed between the same parallel lines and on the same base, and the other is shown to be half the area of a square since they too are both constructed between the same parallel lines and on the same base, and since one of the triangles is half the area of the rectangle, and the other equal triangle is half the area of the square, the square on one side of the right triangle is equal to one of two rectangles into which the square on the hypotenuse is divided. Next, the same strategy is used to show that the other rectangle into which the square on the hypotenuse is divided is equal to the area of the square on the other side of the right triangle, ceteris paribus, because they are double triangles equal to one another. Clearly, one of the deep underlying intuitions is to grasp that two figures can be equal and unequal at the same time, though in different respects: in one respect, they can both have the same area, and yet be different in shape. Another key intuition is to grasp *how* it is that the triangles are half the area of the parallelograms *when constructed between the same parallel lines and on the same base*, and that every parallelogram is of equal area when constructed within the same parallel lines on that same base. For to grasp this idea we have to challenge our intuition that *perimeter is a criterion of area*, which it is not. To grasp this idea, we have to imagine space in such a way that rectilinear figures are imagined within parallel lines—space is conceived fundamentally as a flat surface structured by parallel straight lines, and figures unfold in these articulated spaces. When we imagine in this fashion, we discover that every triangle constructed on the same base, between two parallel lines, *regardless of its perimeter*, will have the same area. Moreover, every parallelogram constructed on the same base as the triangles will have double the area of every triangle, also regardless of its perimeter. To explore this mathematical intuition, we must think through the idea of parallel lines in Euclid's Book I.

To explore this intuition, we should clearly understand that the Pythagorean theorem of I.47 is *equivalent* to the parallel postulate—postulate 5—and this helps us to see that the sequence of proofs from I.27 through I.47 all rely on grasping the character of parallel lines and parallel figures drawn within them. There is no explicit claim that Euclid understood it this way. However, Proclus divides his commentary on the propositions in Book I according to whether the proposition depends on parallel lines or not. That is, the dependence of the Pythagorean theorem on the propositions depending on parallel lines—particularly I.35–41—was understood. By exploring these interwoven interconnections, we can see more clearly the geometrical structure of space for the ancient Greeks.

To identify them both as *equivalent* means that *they mutually imply each other*. To grasp the idea of the parallel postulate, then, is to grasp what follows from the assumption that "If a straight line falls on two straight lines and makes the interior angles on one side less than two right angles, the two straight lines if produced indefinitely meet on *that* side on which are the angles less than two right angles."

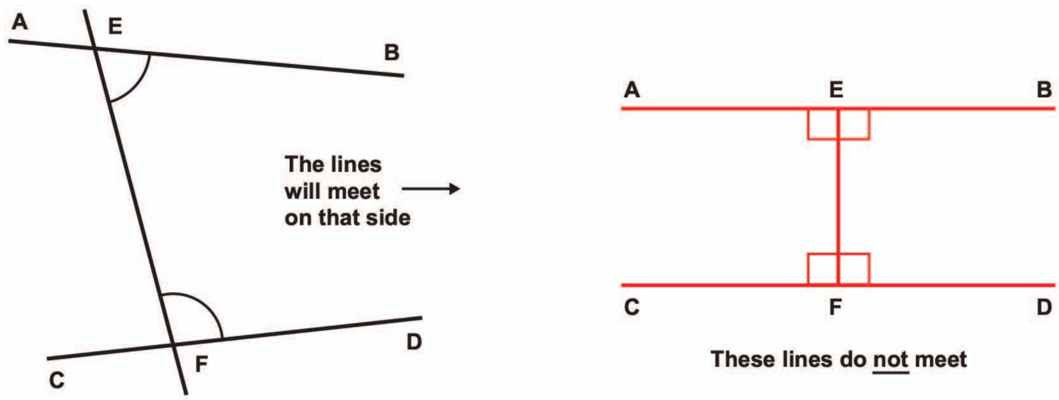

Figure 1.16.

Playfair's axiom is another description of Euclid's fifth postulate; it expresses the same idea by claiming that if there is some line AB and a point C not on AB, there is *exactly one and only one line* that can be drawn through point C parallel to line AB. But it is important to point out that Proclus proposes just this understanding of fifth postulate in his note to Euclid I.29 and I.31.[6]

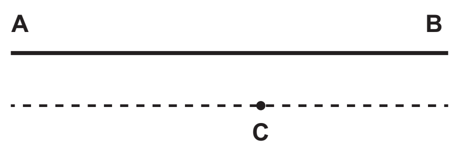

Figure 1.17.

The parallel postulate implies that:

1. In any triangle, the three angles are equal to two right angles
2. In any triangle, each exterior angle equals the sum of the two remote interior angles
3. If two parallel lines are cut by a straight line, the alternate interior angles are equal, and the corresponding angles are equal.

Thus, let us list the parallel theorems of Book I, propositions 27–33:

I.27: If a straight line falling on two straight lines makes the alternate angles equal to one another, the straight lines will be parallel to each other.

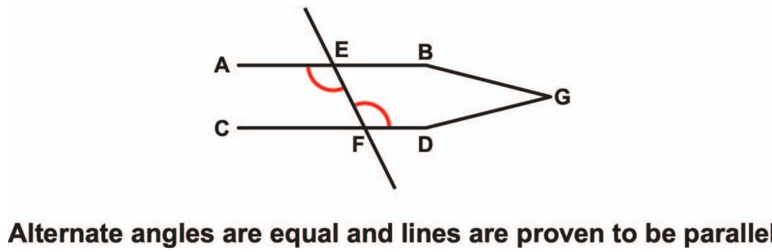

Figure 1.18.

I.28: If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or makes the interior angles on the same side equal to two right angles, the straight lines will be parallel to one other.

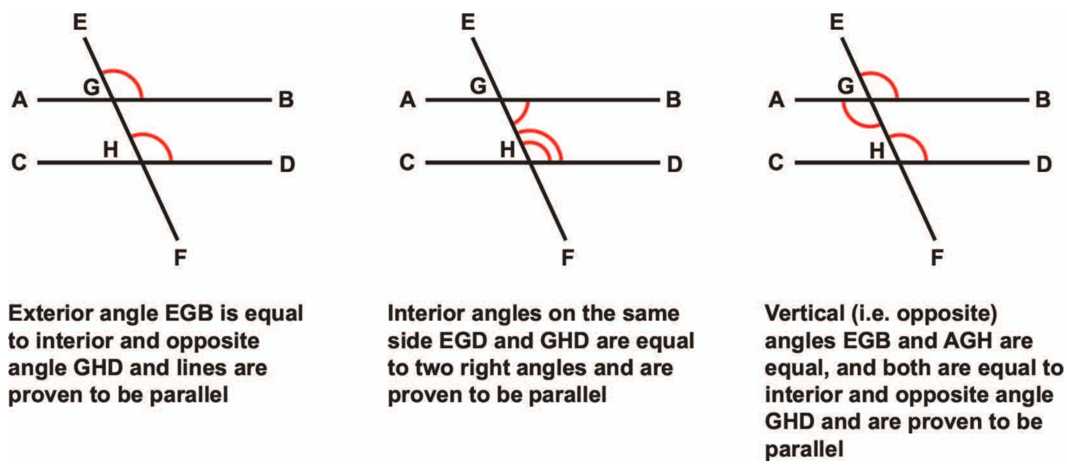

Figure 1.19.

I.29: A straight line falling on a parallel straight line makes the alternate angles equal to one another, makes the exterior angle equal to the interior and opposite angle, and makes the interior angles on the same side equal to two right angles.

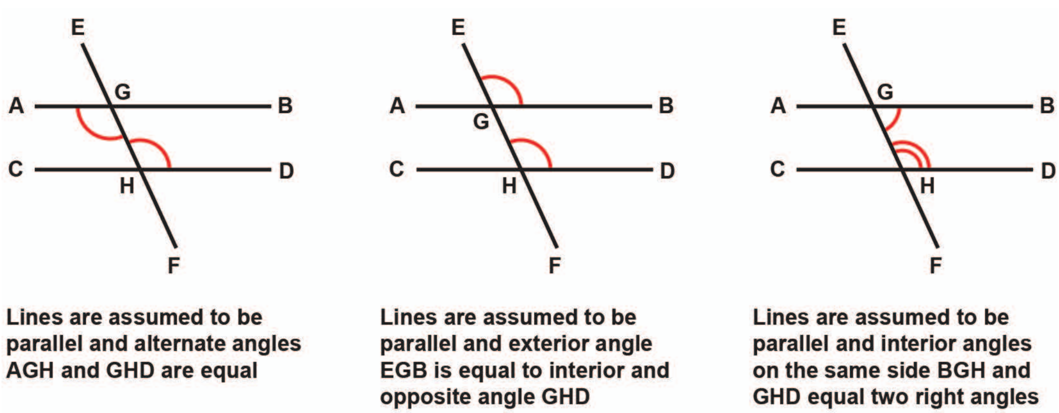

Figure 1.20.

I.30: Straight lines parallel to the same straight line are also parallel to one another.

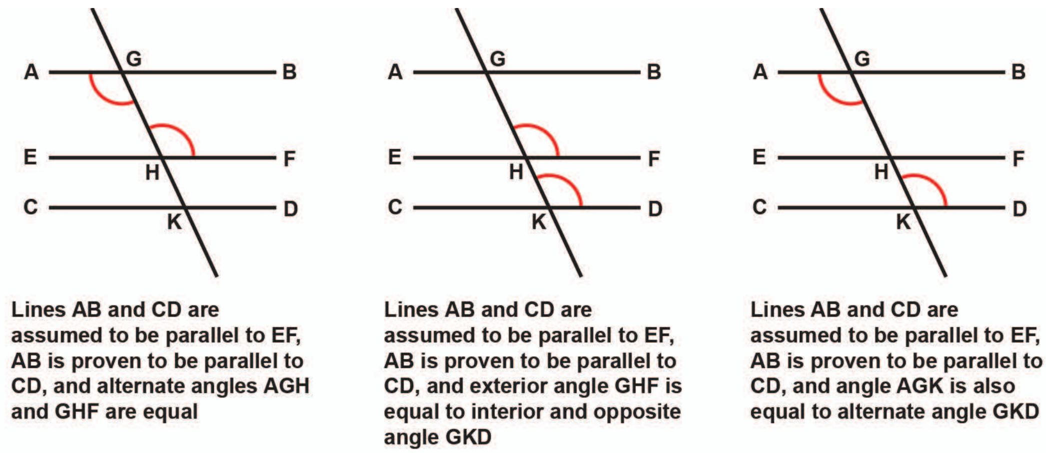

Figure 1.21.

I.31: Through a given point to draw a straight line parallel to a given straight line.

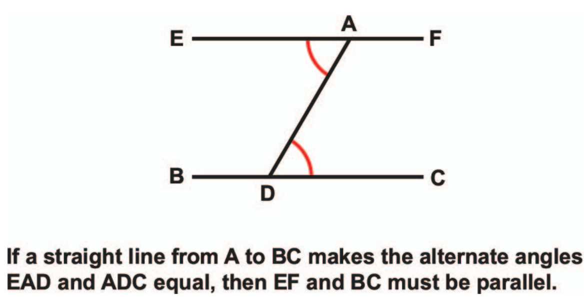

Figure 1.22.

I.32: In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

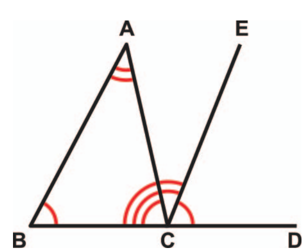

Figure 1.23.

This is the proof, sometimes called "Thales's theorem," that the angles in any triangle sum to two right angles. This is because Thales is also credited with the proof—or some understanding—that every triangle in a (semi-)circle is right, by Diogenes Laertius on the authority of Pamphile, a proof that presupposes the theorem that the base angles of isosceles triangles are equal, and this has been understood to require the prior understanding that the angles of every triangle sum to two right angles. The "proof" of I.32 is credited to the Pythagoreans by Proclus, as we shall discuss in the next chapters, but this does not preclude the likelihood that Thales understood its interconnections; as we considered in the Introduction, working out Geminus's claim that the "ancient" (= Thales and his school) investigated that there were two right angles in each species of triangles—equilateral, isosceles, scalene—means that one can plausibly argue that the principles here would have been understood.

I.33: The straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel.

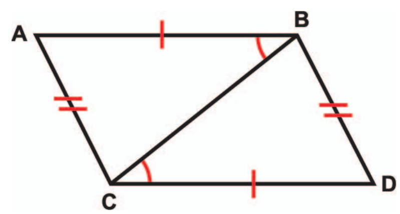

Figure 1.24.

The central idea, for our purposes, is to display the fundamental propositions or theorems that lead to the proof of I.47. They include the parallel propositions and then the areal relations and equivalences between triangles and parallelograms constructed on the same base between the same parallel lines. In setting out theorems I.27–33 we begin to see how Euclid I–VI imagined the world of geometrical objects to be rectilinear on flat surfaces, and that space is imagined as a vast template of *invisible lines*, parallel and not, in the context of which rectilinear figures appear. In that world, parallelograms on the same base are one and all equal to each other, triangles on the same base are also equal to each other, and since every parallelogram can be divided by its diagonal into two equal triangles, every parallelogram has twice the area of every triangle drawn on the same base. To these concerns we, and Euclid, now turn.

Thus, parallelograms on the same base and in the same parallel lines, as in the illustration below, are equal to each other (35, 36) (NB, at I.34, "parallelogrammic *areas*" are first introduced; before this proposition, figures are merely said to be equal, but not equal in area. Again, let us be clear that for Euclid, "area" is not a numerical measure but rather a space contained by the parallelogram. Thus χωρίον is the space contained itself.)

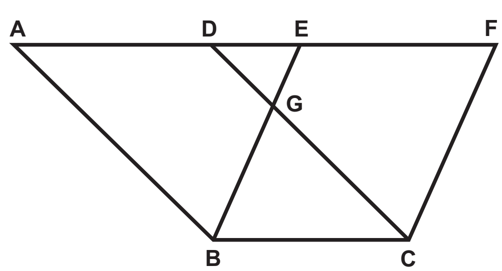

Figure 1.25.

The proof follows the sequence that, first, ABCD and EBCF are both parallelograms on the same base, BC. This means that AD is parallel to BC, EF is also parallel to BC, and since the opposite sides of a parallelogram are equal, AD is equal to EF.

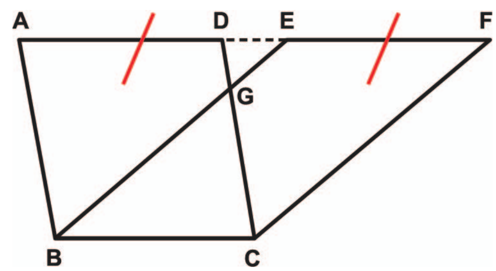

Figure 1.26.

Thus, if DE is added to each, lines AE and DF will be equal. On the other hand, AB is equal to DC, being the opposite sides of parallelogram ABCD. From this it follows from I.4 (SAS equality) that triangle AEB is equal to FDC since sides EA and AB are equal respectively to sides FD and DC, while angle EAB is equal to FDC (AB and DC being parallel).

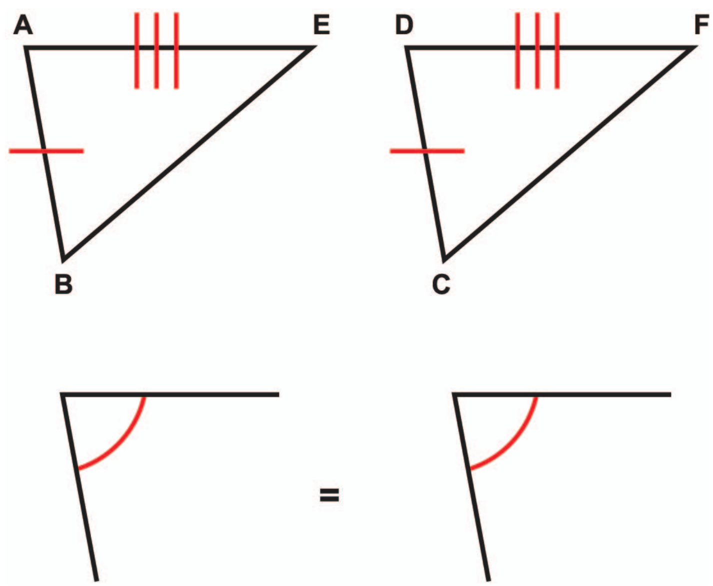

Figure 1.27.

Now, the argument continues, if triangle DGE is subtracted from each, it makes the trapezium ABGD equal to trapezium EGCF, because when equals are subtracted from equals the remainders are equal.

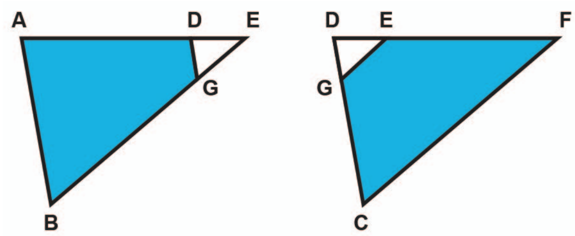

Figure 1.28.

And now, when one adds triangle GBC to both trapezia, the whole parallelogram that results, ABCD, is equal to the whole parallelogram EBCF. Thus, parallelograms on the same bases between the same parallel lines are equal to one another.

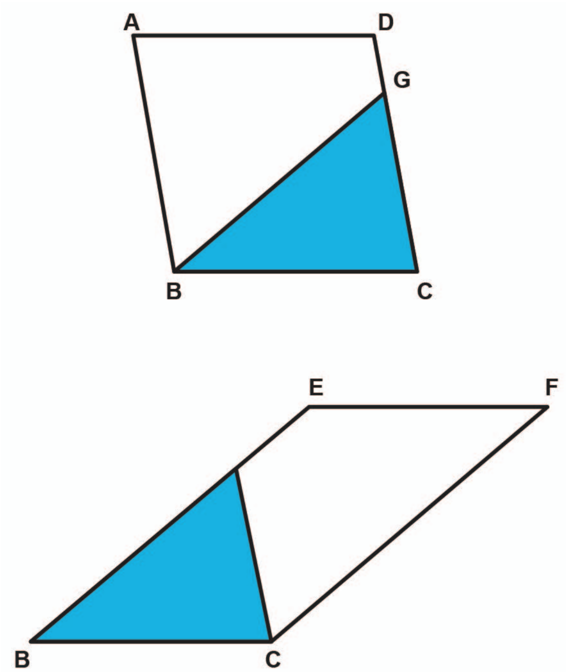

Figure 1.29.

In the next step toward I.47, below, triangles on the same base and between the same parallel lines are equal in area to one another, Euclid I.37:

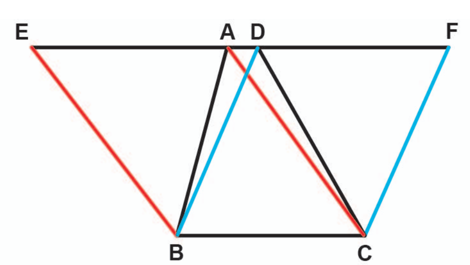

Figure 1.30.

As the diagram at Euclid I.37 makes visible, *triangles on the same bases and in the same parallels have equal area.* Euclid has already established that parallelograms have opposite sides equal, and that parallelograms on the same base and in the same parallel lines are equal in area to each other. Now he extends these arguments to triangles. He places two triangles, BAC and BDC, on the same base, BC, and then creates a parallelogram out of each, EBCA and DBCF, and each of them is equal to the other, since they too are on the same base and between the same parallel lines.

Since the triangles bisect the equal parallelograms—AB bisects EBCA, and FB bisects DBCF—the triangles must be equal because each is half the area of the equal parallelograms.

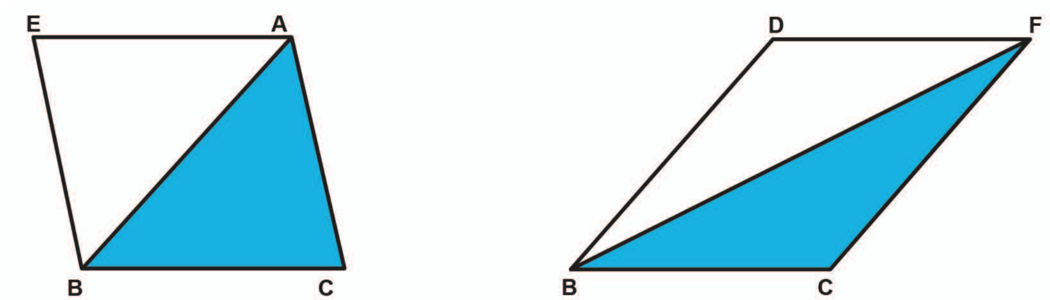

Figure 1.31.

Thus the proof that triangles on the same bases and in the same parallel lines are equal to each other follows from the proof for areal equivalence between parallelograms. *Note that every parallelogram can be imagined as being built out of two triangles, created by a diameter—a diagonal—that bisects it, and thus every parallelogram is imagined as being dissectible into triangles. The deep intuition is that all parallelograms reduce ultimately to triangles, and that the relation between triangles can be illuminated by projecting their areas into parallelograms composed of them.*

And then, finally for grasping the sequence in I.47, we have I.41, where Euclid proves that every parallelogram that shares the same base as any and every triangle, within the same parallel lines, must always have double the area of any and every triangle.

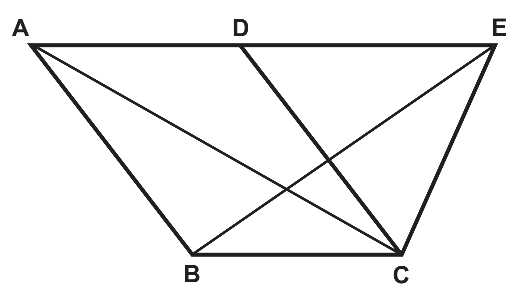

Figure 1.32.