On the Universality of Human Mathematics

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1.0. Introduction

It has often been stated that mathematics would serve as a universal language, one suitable for communication between totally alien societies. Our purpose here is to examine that statement in detail. We shall see that while mathematics is often motivated by scientific applications, it is equally likely to arise from internal sources, sources that have nothing to do with the world of science. Nevertheless, we argue that human mathematics can be understood by any race that has a science, and can be an effective means of mutual communication. There are a number of "philosophies" of mathematics (Pinter 1971) but, in this connection, the views of only two of these need concern us: Extreme Platonists and Strict Formalists. The difference between them is apparent in how they answer the following question: Are the natural numbers, 1, 2, 3, 4,… merely creations of the human mind or do they exist independently of us? The Platonic view is that these objects, and indeed all mathematical objects, really exist, perhaps in some hyperworld. In this view the mathematician is rather like the scientist. He, or she, discovers objects that are "out there." So if an alien intelligence exists then they, too, could discover the same mathematical objects that we have found, for instance, real and complex numbers, functions, topological spaces, etc.

The strict formalist, however, has a very different view. To her, or him, mathematics is a kind of game played by specific rules. Somewhat like chess. An unsolved problem gives a kind of goal, and solving such a problem constitutes a "win." So, to a strict formalist, while five fingers, five cars, and five dollars certainly exist, the number 5 does not. It is a creation of the human mind and an alien, however intelligent, might have no knowledge of 5 or of any other human mathematical object.

My own position is a strange combination of the two. I think the natural numbers do exist independently of us. The rest of mathematics, however, might not exist anywhere but in our minds. But since, as we shall see, all of mathematics can be based on the notion of natural number, all of our mathematics could, in principle, be communicated to any intelligent alien who understands these numbers, certainly to any race whose members can count. It will become apparent, however, that the world of mathematics is not the world of physical reality. It is an artificial world, a world of abstractions and idealizations that human mathematicians have created over many centuries. It may be more reflective of our minds than we realize and may say more about human nature than it does about the real world. Still, one must not forget that human mathematics has an uncanny habit of becoming useful either in explaining some aspect of reality or modeling that reality (Dantzig 2007). So as strange as it might appear to an alien he, or she, or it will be able to appreciate its value.

2.0. Geometric Problems

Humankind has been aware, for many millennia, of the fact that space has properties and that these properties could be usefully exploited. The annual flooding of the Nile forced the ancient Egyptians to find ways to correctly reset the boundaries between adjacent farms, and this involved some geometric insight. As early as 1700 BC, the peoples of Mesopotamia knew about the Pythagorean Theorem, a rather sophisticated piece of information for such an early civilization (Edwards 1984). In the nineteenth century, when it was believed that the moon was inhabited, the mathematician Gauss suggested we use this theorem as a basis of a message to our lunar neighbors. It was the ancient Greeks, however, who first systematically studied space and gave us an organized geometry. This has come down to us in the books of Euclid written about 300 BC. His approach was axiomatic (Pinter 1971). From a few simple assumptions, he deduced all that was then known. This provided us with an ideal, a canonical model, to which all succeeding generations of mathematicians aspired, and there were attempts to base all of mathematics on geometry; numbers, for example, were to be thought of as geometric ratios. Geometry is, of course, a major branch of modern mathematics with sometimes striking developments occurring centuries apart. In the 1600s it was combined with algebra, by Descartes and Fermat (Dantzig 2007), to give us analytic geometry, a subject that enabled us to bring our considerable geometric insight to bear on problems of algebra, and to use algebraic techniques to solve problems of geometry. With the advent of calculus in the seventeenth century mathematicians applied these ideas to the study of curves and surfaces, giving us differential geometry. The subject blossomed in the nineteenth century with the surface theory of Gauss, the brilliant insights of Riemann (Dantzig 2007), and the tensor calculus of Ricci and Levi-Civita (Adler, Bazin, and Schiffer 1965). This was the mathematics drawn upon by Einstein to formulate his far-reaching general theory of relativity. It is no diminution of his genius to point out that the mathematics was there, fully developed by those whose interests were purely mathematical, for him to use. This happens far more often than is generally recognized; mathematics is developed for purely mathematical reasons long before it finds application to some area of science. This is an important point. It shows that human reasoning is an effective tool for understanding physical reality. As Galileo insisted, we can understand the world and unravel the mysteries that surround us (Hawking 1988). This understanding does not come only from mathematics, of course, but from the work and insights of physicists, chemists, geologists, etc. Communication with an alien race should be possible, at least up to a point, if that race has a science; because science is a study of physical reality and we share the same physical reality (DeVito and Oehrle 1990). Geometry is by no means a dead subject. There were remarkable developments in the twentieth century and research continues today.

Geometric problems led to other branches of mathematics even in ancient times. Due to an unfortunate series of incidents, Dido, a Phoenician princess, needed to relocate to Africa. But those already there were reluctant to sell her land. Finally, after the exchange of a great deal of money, it was agreed that she could have all the land she could enclose in an ox hide. A bad deal it would seem. But Dido was clever and resourceful. She cut the ox hide into thin strips, tied the strips together to make a rope and demanded all of the land she could encompass with the rope. Here we have a problem, what shape encloses, in a fixed perimeter, the most area. To see that there really is a problem here, note that with 100 ft., say, of rope you can encompass a rectangle with two sides 49 ft. and two sides 1 ft. each, giving you an area of 49 ft^2. But with the same 100 ft. rope you could also enclose a square having each side 25 ft., giving you an area of 625 ft^2., considerably more land. There is a branch of mathematics that deals with problems of this kind (van Brunt). It is called the calculus of variations. The subject was developed in the eighteenth century by Euler, Lagrange, and the Bernoullis and now has numerous applications to physics, such as Fermat's principle of optics, Hamilton's work in mechanics, etc. The answer to Dido's problem turns out to be a circle. This curve encloses the maximum area for a given perimeter. According to legend, Dido used her rope to enclose a large circle, claimed the land within it, and founded the city of Carthage upon it.

3.0. Numerical Problems

Humans had a sense of numerical relationships long before numbers themselves were discovered. If the elders of a prehistoric clan wanted to know if they had enough spears to equip a hunting party all they had to do was have each man pick up a spear. If all the men were armed and there were still spears left on the ground, they had plenty of weapons; there were more spears than men since the men were in one-to-one correspondence with a sub-collection of the spears. On the other hand, if all the spears were taken and some men were left emptyhanded then they clearly had an equipment shortage. There were more men than spears because here the spears were in one-to-one correspondence with a sub-collection of the men. The final case, of course, is when each man is armed and there are no spears left over. In this case the two collections, men and spears, are equinumerous, because there is a one-to-one correspondence between them. This idea was used by ancient peoples all over the world to keep track of their herds or even their armies. They set up one-to-one correspondences between the collection they wanted to keep track of and notches on a stick or pebbles in a pile. No counting is involved here. Counting is a rather sophisticated process, and to begin doing it we must first have a standard set of models for our numbers. The wings of a bird gave a natural model for the number two, the fingers on a hand gives a model for the number five, and other models suggest themselves gradually giving rise to the concepts of two, five, etc. Here, however, the numbers are seen as cardinals, the number of elements of a given collection. To begin counting we need to discover the ordinal aspect of numbers; the fact that they form an ordered sequence with each number occupying a specific place. When we count a collection of objects we mentally label them as first, second, third, fourth, and, let's say, fifth, and we conclude that there are five objects in our collection. We pass from ordinal number to cardinal number with such ease that we are rarely aware of it. I strongly suspect that the ideas outlined here arose from human observation of the day-night cycle and the realization that that cycle could be usefully exploited. In keeping track of a herd you could lead the animals one by one past a carver, a person who would record the passing by carving a notch on a stick. The order here is unimportant. You choose the animals randomly. But in keeping track of the days needed for a particular journey, this is what was important for early man, not distance, which he couldn't measure anyway; you had to record the days as they came, in their natural sequence. This is what may have led to the idea of ordinal number and its relation to cardinal. It wasn't easy, and it took time for the idea to crystallize, but eventually humans realized that ordinal numbers were important because they gave the only practical way of finding the number that really interested them, the cardinal number of a collection. The natural numbers may exist, waiting to be discovered by us or by some alien race. But we were led to this discovery by the presence on our planet of a day-night cycle. This astronomical property of a planet may be the trigger needed to lead the inhabitants of that planet to this important discovery.

4.0. Infinity

The term *infinity* is the source of much confusion and general misunderstanding. In elementary calculus it is used as a shorthand, a "way of talking." The numbers ½, 2/3, 3/4, 4/5…increase, as is obvious from their decimal representations 0.5, 0.666…, 0.75, 0.8…but they never exceed the number 1. We say they have an upper bound. A sequence of numbers that increases without an upper bound, like the sequence 2, 4, 8, 16, 32, 64…, is said to "tend to infinity." A sequence that tends to infinity is simply a sequence that increases without an upper bound. There is a geometric interpretation of this (Conway 1978). In function theory the complex plane is mapped stereographically onto a sphere, called the Riemann sphere. You set the sphere on the plane with its' "south pole" at the origin and, given a point in the plane, you draw a line from the "north pole" of the sphere to the given point. The point where this line intersects the sphere is taken as the image on the sphere of the given point. All points of the plane are, in this way, put in one-to-one correspondence with points on the sphere, but the north pole of the sphere corresponds to no point in the plane. This is called the "point at infinity" and a sequence of points in the plane whose distance from the origin increase without bound is said to converge to infinity; the corresponding points on the sphere tend toward the north pole.

In classical mathematics a distinction was made between a collection that was actually infinite, in which infinitely many objects are thought of as existing simultaneously, and a collection that was potentially infinite (Pinter 1971). The collection of all natural numbers, for example, was potentially infinite. No matter how many of these numbers we write down, we are aware that there are always more. It was Georg Cantor (1845–1918) who, while working on a problem in the technical area of trigonometric series, was led to consider collections that were actually infinite. This was very disturbing to the mathematicians of his time and led to much criticism. The idea of a "set" was making its way into mathematics; it is a remarkably useful idea and leads to an underlying unity that is elegant and insightful. Cantor's work cast some doubt on the wisdom of relying too heavily on this concept. Today, mathematicians distinguish two types of set theory: axiomatic and naïve. In the latter a set is any well-defined collection of objects. Well-defined means that it must be clear just what objects are in the set, just what its members or "elements" are, and just what objects are not in the set. To Cantor the collection of *all* natural numbers, 1, 2, 3, 4…, is a perfectly good set even though it contains infinitely many elements.

A good deal of Cantor's early work consisted of setting up one-to-one correspondences between various sets, something that, as we have seen, humans have been doing for millennia. But applying this idea to infinite sets led to some counterintuitive, and even disturbing, problems. The first surprise was already noted by Galileo (Dantzig 2007). In the course of his "Dialogs Concerning the New Sciences," the question arises as to whether there are more squares, 1,4,9,16,25,…, or more natural numbers, 1,2,3,4,5,… . One of the characters notes that every square is a natural number, meaning that the set of squares is a subset of the set of natural numbers, but there are many numbers that are not squares, for example, 2,3,5,6,… . Thus, the squares are a proper subset of the natural numbers and hence contain fewer members. A second character, however, points out that the two sets can be placed in one-to-one correspondence: 1 corresponds to 1, 2 corresponds to 4, 3 corresponds to 9, etc. Galileo's conclusion is that all one can say is that both sets are infinite; neither is larger than the other. Cantor would say that the two sets are equinumerous. This is, in fact, a characteristic property of infinite sets: A set is infinite if, and only if, it can be placed in one-to-one correspondence with one of its proper subsets (Dantzig 2007; Hrbacek and Jech 1984).

The second surprise is, perhaps, even more striking. Cantor tried to set up a one-to-one correspondence between the set of natural numbers and the set of real numbers, the so-called continuum. He found that no such correspondence exists. There are more real numbers than there are natural numbers even though the two sets are both infinite! So there are "degrees" or "orders" of infinity. Some sets are infinite, but some sets are even "more infinite." There is, in fact, no limit to the infinities. Given any set, one can show that the collection of all of its subsets (sets whose elements are also in the original set) is larger than the given set. If you start with an infinite set and collect all of its subsets into a new set, the new set is more infinite than the one you started with (Enderton 1977).

The results of set theory and our ideas on geometry, outlined above, combine to show forcefully that the world of mathematics is not the same as the world of physical reality. Using a technical result due to Hausdorff, Stefan Banach and Alfred Tarski showed that one can cut a sphere the size of a pea into a finite number of pieces, reassemble the pieces and obtain a sphere the size of the Sun (Wapner 2005)!

5.0. Putting It All Together

It has long been the goal of mathematicians to unify the entire subject; to deduce all of mathematics from a small number of basic axioms. We have mentioned that attempts were made to base all of mathematics on geometry, but these never got very far. The work of Karl Weierstrass, Richard Dedekind, and others in the nineteenth century, showed that all of mathematics could be based on the notion of natural number (Dantzig 2007). Since, in my view, these numbers will be known to any race that has the radio telescope, I think that all of human mathematics can be communicated to such a race. This does not mean that they will have arrived at our mathematics themselves. They may think very differently and go off in directions we cannot anticipate. Just as alien chemists, using the same chemical elements we know here, might go off into aspects of chemistry our chemists would never think of; not because they aren't smart enough, but because they are human and hence have the same limitations as all other humans, limitations the aliens might not have. But whatever they do will be understandable to earth chemists, and aliens and humans should be able to understand each others' mathematics insofar as these are based on natural number.

Cantor's work, because of its elegance and inherent beauty, began to be accepted and incorporated into mathematics when, ironically, certain disturbing paradoxes arose in connection with it. Some of these were logical, some were semantic. The latter led mathematicians to study formal languages. Some of the early computer languages were based on this work and Hans Freudenthal drew on these ideas when he developed Lincos, a language specifically designed for cosmic communication.

Let's sample two of these paradoxes here. Perhaps the best known of the logical paradoxes is that due to Bertrand Russell (Enderton 1977). We can present this in ordinary language, avoiding technical jargon. There is a town so small that it has only one barber, and he shaves those men, and only those men, who do not shave themselves. Now, who shaves the barber? If he does not shave himself then he must be shaved by the barber. But he is the barber. So if he does not shave himself, then he does shave himself—a clear contradiction. Well, maybe he does shave himself. But then, as a man who shaves himself, he is not shaved by the barber—another contradiction. At this time, and even now, people were working with sets of sets, like a league, which is a set of teams each of which is a set of people. So Russell considered the set of those sets that were not members of themselves (the set of all books is not a book, but the set of ideas is an idea). This set leads to the same contradictions that the barber led us to.

An interesting semantic paradox is the one due to Berry (Enderton 1977). It is a deep and useful fact that any non-empty set of natural numbers has a smallest member. The smallest odd prime is 3, the smallest perfect number is 6, and the smallest seven digit number is 1,000,000. Now suppose we choose a dictionary and consider all numbers that can be described in twenty-five or fewer words from that dictionary. There are only finitely many words in the dictionary and only finitely many ways of combining these words into sentences of twenty-five words or less, so the set of numbers that cannot be described in this way is non-empty. So it has a smallest member, say, N.

Then: "N is the smallest number that cannot be described in twenty-five words or less from our dictionary." But the sentence in quotes describes this number in just eighteen words!

Some wanted to reject the whole of set theory because of these paradoxes, but many others felt that that was going too far. In 1900, an international congress of mathematicians was held at the University of Paris. Here one of the greatest mathematicians then active, a young German, gave a list of problems to challenge the mathematicians of the coming century. His very first problem involved Cantor's work, thereby lending his support to Cantor and to the importance of set theory. The young German was, of course, David Hilbert, and the problem he posed was this: We have seen that the set of real numbers, the continuum, is larger than the set of natural numbers. Is there an infinite set that is between these two, that is, a set that is larger than the natural numbers but smaller than the continuum? It was conjectured that this could not be and this conjecture was shown to be consistent with the other axioms of set theory (see below) by Kurt Gödel in the 1930s. To the surprise of all, in 1963, Paul Cohen showed that the negation of this conjecture was also consistent with the axioms of set theory. Here is a statement, "There is no set strictly larger than the natural numbers but strictly smaller than the continuum," that cannot be proved and cannot be disproved! Gödel had shown that, if mathematics is consistent, then statements of this type, not provable and not disprovable, had to exist, but no one even imagined that the continuum problem was such a statement (Enderton 1977).

The paradoxes of set theory led mathematicians to try to formulate this subject in terms of a collection of axioms that, it was hoped, would avoid these paradoxes and allow us to keep the useful aspects of that subject. Perhaps the best known are those of Zermelo and Fraenkel; let's call this ZF. But one more axiom had to be added to this collection, the so-called axiom of choice; let's call that AC (Hrbacek and Jech 1984). When first stated, this axiom seems harmless, even obvious. But it has far-reaching consequences (Hrbacek and Jech 1984). The result of Banach and Tarski, stated above, relies on it as do many other results of analysis, topology, and even modern algebra. Let me give a whimsical illustration of the AC. Imagine a forest in which infinitely squirrels live. Each squirrel has its own tree and hidden in that tree has a hoard containing infinitely many acorns. Does it not follow that there are more acorns than squirrels? To pin this down we would have to set up a one-to-one correspondence between the squirrels and some subset of the acorns. This would be easy if we could assign to each squirrel one of the acorns in its hoard. But how would you select that acorn? What criterion would you use? You can't do it one at a time because there are infinitely many squirrels and you'd never finish. The only way to do this is to give some rule whereby each squirrel can select an acorn in its hoard.

But it is not at all clear how to do this—can you interview infinitely many squirrels and find out which acorns they would select? The AC says that for any collection of non-empty sets there is a rule whereby you can assign to each set one of its members. The axiom gives no clue as to how one can do this. How would you select from each galaxy a particular star in that galaxy? Even an astronomer might have trouble coming up with a rule for doing this and the set of galaxies is finite.

It is now known that the AC is independent of the other axioms of ZF and that if ZF is consistent, then adding this axiom does not produce any inconsistency (Enderton 1977). I should mention that there is a small set of axioms, the so-called Peano Postulates (Hrbacek and Jech 1984), from which all of the properties of the natural numbers can be deduced. There is no need to go all the way back to set theory in our communication of human mathematics to an alien race. Perhaps it is the human fascination with the infinite that attracted us to the direction pioneered by Cantor and maybe it is that same fascination that keeps us engaged in set theory and the foundations of mathematics.

6.0. Conclusion

I have taken the natural numbers as the set 1, 2, 3…, which is the same as the set of positive integers (in some books the natural numbers start with 0). I think these numbers exist and are, or could be, known to any intelligent race. The rest of mathematics is, to my mind, a human creation. As Kronecker might have said it, "God made the positive integers, all the rest is man's work" (Dantzig 2007). Mathematics is as much a part of us as is our Music and our Art. Aliens may not share any of it, but they should be able to understand it because it can all be based on natural numbers. To an alien, human mathematics might appear curious and exotic. It may give them insights into physical reality that they could never come to on their own, just as their mathematics might give us such insights. Here is one aspect of humanity that can be communicated to any society whose members can count. What deductions they will make about human nature from our mathematics is impossible to predict. Our extensive study of space, our geometry, may stem from our reliance on the sense of sight. Our study of the mathematics of motion, calculus, may stem from our having played the role of prey and that of predator. When something is chasing you, you have no time to think about how to run or climb. These things must be internalized. Similarly, you cannot think about how to aim your spear at a fleeing animal; you must "know" what to do. This might be why many find calculus very natural and comfortable. But even here our sense of sight enters in. In calculus we seek to approximate a given curve at each point by a line. The derivative at that point gives us the slope of that line. Would an alien race with a different evolution think this is natural? Should we choose to discuss set theory, the human preoccupation with the infinite will soon become apparent. Would an alien race share this with us also, or would they find it strange, even weird? The mathematics of an alien race may, in a similar way, tell us a great deal about that race. Surely we would like to learn all we can about any alien race we contact. The realities of this kind of interaction, however, severely limit what we can hope to learn, at least in the near term. The main advantage of mathematics over other subjects is that, in the context of alien-human communication, it has a good chance of being understood.

Works Cited

Adler, R., M. Bazin, and M. Schiffer. 1965. *Introduction to general relativity*. New York: McGraw-Hill.

Conway, J. B. 1978. *Functions of one complex variable*. New York: Springer-Verlag.

Dantzig, T. 2007. *Number: The language of science*. New York: Penguin Group.

DeVito, C., and R. Oehrle. 1990. A language based on the fundamental facts of science. *Journal of the British Interplanetary Society* 43: 561–68.

Edwards, H. M. 1984. *Galois theory*. New York: Springer-Verlag.

Enderton, H. B. 1977. *Elements of set theory*. New York: Academic Press.

Freudenthal, H. 1960. *Lincos: Design of a language for cosmic intercourse*. Amsterdam: North Holland Press.

Hawking, S. W. 1988. *A brief history of time*. New York: Bantam.

Hrbacek, K., and T. Jech. 1984. *Introduction to set theory*. New York: Marcel Dekker.

Pinter, C. C. 1971. *Set theory*. London: Addison-Wesley.

Van Brunt, B. 2004. *The calculus of variations*. New York: Springer-Verlag.

Wapner, L. M. 2005. *The pea and the sun*. Wellesley, MA: A. K. Peters.