INTRODUCTION

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HISTORICAL PRECEDENTS

The interest in compiling a book of recent research on the development of multiplicative concepts evolved from a variety of efforts. Research on multiplication and division revived in the early 1980s with the work of Fischbein, Deri, Nello, and Marino (1985) who hypothesized that "Each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model" and argued that these models impose constraints on students' predictions of the operation needed when solving multiplication and division with different decimal numbers. They conjectured that the primitive intuitive model for multiplication was repeated addition and for division was based in either partitioning (sharing) or repeated subtraction. This work built on and was further complemented by a line of research by Bell, Swan, and Taylor (1981) and subsequently by Bell, Fischbein, and Greer (1984), Luke (1988), and Graeber and Tirosh (1988)—all of whom investigated how these models would lead to the assumption that multiplication makes bigger and division makes smaller. Recent work has examined further the impact of numeric form and value on students' and teachers' selection of operation and is reported in this book (Harel, Behr, Post, and Lesh, Chapter 10). An issue of *The Journal of Mathematical Behavior* (vol. 7, no. 3, December 1988) with guest editor B. Greer was devoted to the topic.

In 1983, in the *Acquisition of Mathematics Concepts and Processes* (R. Lesh and M. Landau, editors), the research activities of two groups set the stage for a second wave of research in this arena. Vergnaud introduced to an American audience the idea that a conceptual field is a "set of problems and situations for the treatment of which concepts, proce-

dures and representations of different but narrowly interconnected types are necessary" (p. 127). He discussed the treatment of a multiplicative conceptual field as an example and identified the broad strands of this conceptual field to include: multiplication, division, fractions, ratio, rational number, linear and *n*-linear functions, dimensional analysis, and vector spaces.

In the same volume, results from the Rational Number Project were reported by Behr, Lesh, Post, and Silver. By reviewing and analyzing the seminal work of such scholars as Dienes, Karplus, Kieran, and Hart, they offered a synthesis of the field and identified six "subconstructs" of rational number: part to whole comparison, decimal, ratio, indicated division (quotient), operator, and measure of continuous or discrete quantities. These two papers established a common theme. It was no longer sufficient to analyze the cognitive development of these ideas in isolation. There was a recognition that the ideas were interwoven into a field of related concepts, whose acquisition would not be linear or piece by piece. As with a spider's web, contact with one strand would reverberate across the entire space.

In 1986, a series of conferences were held reviewing significant bodies of research under National Science Foundation's Research Agenda Project for Mathematics Education (Sowder, 1988). One of these groups produced *Number Concepts and Operations in the Middle Grades* (J. Hiebert and M. Behr, editors). The majority of these papers treated topics from the multiplicative conceptual field. This collection provided a case for rethinking the development of the number concept through the grades. As the editors put it:

> Mastery of many of the number concepts and number relationships in the middle grades appears to require a reconceptualization of number, a significant change from the primary grades in the way number is conceived. . . . Given the fundamental changes in the nature of number . . . , it is not surprising that significant cognitive reorientations are needed to construct and comprehend such changes. This means that it is likely that there are not smooth continuous paths from early addition and subtraction to multiplication and division, nor from whole numbers to rational numbers. Multiplication is not simply repeated addition, and rational numbers are not simply ordered pairs of whole numbers. The new concepts are not the sums of previous ones. Com-

> petence with middle school number concepts requires a break with simpler concepts of the past, and a reconceptualization of number itself. (Hiebert and Behr, 1988, p. 8)

In addition to work by the researchers just cited, the book included such work as that conducted by Nesher, which introduced an instructional dimension into the discussion of models of multiplication. Nesher made distinctions between "mapping rule" multiplication (three books per shelf and four shelves), multiplicative compare problems (three times as many this as that), and Cartesian multiplication (four shirts, three pants, how many outfits?). She connected these distinctions to distinctions made by Schwartz (1988) between different types of quantity and multiplication of these various types of quantity by each other: intensive by extensive, scalar by extensive, and extensive by extensive. Nesher reported that students' success in solving multiplication problems seemed heavily dependent on the instructional setting and linguistic cues.

A follow-up conference to the Research Agenda Project meeting was held in San Diego, and a subgroup was formed to pursue further the ideas of the "multiplicative conceptual field." With support from the National Center for Research in Mathematical Sciences Education and the National Science Foundation, this group met four times over the period of 1989–1990, chaired by the editors of this book. The chapters in this volume are, for the most part, a result of those meetings and discussions. A few were solicited separately when their authors were unable to attend the meetings. The editors wish to acknowledge the contributions of Kenneth Carlstrom to publication of this book. His careful reviews and comments contributed significantly to improvement of the manuscript. Support for these meetings and manuscript preparation came from NSF grant MDR 9053590.

SHARED COMMITMENTS

The contributors to this monograph share four basic commitments:

1. *The topics included under the rubric "multiplicative conceptual field" (MCF), possess an interconnectedness and complexity that presents a unique research challenge.* This

sense of connectedness is repeatedly expressed throughout the volume: "[R]esearchers should study conceptual fields and not isolated situations or isolated concepts" (Vergnaud, p. 46); "The domain represents a critical juncture at which many types of mathematical knowledge are called into play, and a point beyond which a student's understanding in the mathematical sciences will be greatly hampered if the conceptual coordination of all the contributing domains is not attained." (Lamon, p. 90);

2. *The ideas of the MCF develop over considerable time periods, and new topics require the reconsideration of old topics as one develops a mature understanding of the field.* A developmental approach makes it essential for researchers of elementary, middle and secondary levels to discuss, debate and share their insights. "[O]ne of the goals of current research is to identify important mathematical processes, themes, or connections by which thinking becomes progressively more sophisticated from early childhood through early adulthood" (Lamon, p. 90). "This chapter is concerned with the extension of meaning of multiplication and division from their early conceptualizations" (Greer, p. 61). "Such guidance, necessarily, would have to start from points that are accessible to the child; and to establish these starting points it seems indispensable to gain some insight into the child's conceptual structures and methods, no matter how wayward or ineffective they might seem" (Steffe, p. 5).

3. *Among the authors there is a willingness to think deeply about and to rethink and revise the epistemological content of the area.* "We are proposing an alternative experiential basis for the construction of number in a primitive cognitive scheme we label *splitting*" (Confrey, p. 291). "In undertaking historical analysis, we are not advocating the naive view that individual learning should follow historical development, but rather looking for ways in which a 'rational reconstruction' . . . of the historical genesis of mathematical concepts may complement our work with students, helping us see students' work from a different perspective" (Smith and Confrey, p. 332). "We will now examine the several forms of competent, but informal, reasoning that have been commonly observed in missing-value problem situations, and then contrast these with the more formal equation-building approach typically taught in schools" (Kaput and West, p. 245). "In our recent effort to better understand the multiplicative concep-

tual field, and in particular the transition phase from the additive structure to the multiplicative structure, we probed into [the question of incongruity between the meaning of] multiplication and division in the whole number domain [and that in] the rational number domain" (Harel, Behr, Post, and Lesh, p. 363). "This perspective calls for deep, careful, and detailed analysis of mathematical constructs both to exhibit their mathematical structure and to hypothesize about the cognitive structures necessary for understanding them. Such analysis would lead to a theory about mathematical knowing and learning that could guide cognitive research" (Behr, Harel, Post, and Lesh, p. 124).

4. *To learn about a multiplicative conceptual field, one must examine its relationship to the situations in which multiplicative reasoning occurs and not view its ideas as isolated abstractions.* "The premise behind this investigation is that when people reason mathematically about situations, they are reasoning about *things* and *relationships*. The 'things' reasoned about are not objects of direct experience and they are not abstract mathematical entities. They are objects derived from experience—objects that have been constituted conceptually to have qualities that we call *mathematical*" (Thompson, p. 180).

THE CONTENTS

The need for a new publication on the topic was due to the progress that has been made in this field. As one reads through this volume, it becomes apparent that certain issues concerning units, ratios, rates, and recursion are emerging as fundamental ones. These are not treated uniformly in the book, but they continually resurface as explanatory concepts for the research reported. A second reason for the production of a new volume is that the topics included in MCF seem increasingly critical in school mathematics:

1. Mullis, Dossey, Owens, and Phillips (1990) report results that suggest that these topics are still poorly learned. At grade 8 only 49 percent of the students answered the following question correctly: "The weight of an object on the Moon is 1/6 the weight of that object on the Earth. An object that weighs 30 pounds on Earth would weigh how many pounds on the Moon? Answer 5 " (p. 476).

2. These topics create the critical juncture in middle school, separating those students who persist and those who drop out.

3. There is evidence that these topics are poorly understood by elementary teachers, and hence, effective methods for approaching them will have an impact not only on students but on teachers (Harel, Behr, Post, and Lesh, Chapter 10 in this volume; Simon and Blume, 1992).

4. These topics are critical in technological environments, where decimal notation typically replaces ratio and root displays and where issues of scaling require a deep understanding of the rational numbers and operations.

5. These topics provide a rich and fertile source of problems as mathematics educators recognize the need for increased use of situations, contexts, and problem solving.

Shulman (1978) argued for the importance of critical research sites. A critical research site is one in which the production of new insights will lead to dramatic reconceptualization in both that arena and others. All too often researchers choose topics whose primary criterion for selection is their amenability to available research methods, using the field's current concepts and forms of methodology, an approach which slows down progress in the field.

In recognizing the complexity of the topics in MCF, the work reported in this volume represents a rather substantial departure from traditional mathematics education research. The research projects represent an attempt to consider seriously the difficulties and challenges students and teachers face when approaching the topics of multiplication, division, ratio and proportion and to find new ways to think and talk about those difficulties and challenges. In this volume, to create some of those new descriptions, the contributors have moved beyond the presentation of particular research results to articulate and examine their conceptual and methodological commitments, placing their results in a larger framework.

Part I. Theoretical Approaches

In the first part, the authors seek to establish a conceptual paradigm for their work and provide examples of what is produced from such a paradigm. Leslie Steffe locates his work in

radical constructivism. He contrasts Piaget's claim that the operations of the child may not mirror adult operations in any simple way, with "The current notion of school mathematics [that] is based almost exclusively on formal mathematical procedures and concepts that, of their nature, are very remote from the conceptual world of the children who are to learn them" (Steffe, p. 5). Steffe calls for very close attention to be paid to the current structures of the child and argues for the teaching experiment as the most viable way to allow sustained interactions in which a child's initial organizations and operations can be inferred and the development of new schemes can be observed.

In contrast, Vergnaud begins from a different starting point. Rejecting the trends in psychology to explain all cognition from non-discipline-based positions (information processing, and so on) he argues for subject-matter specificity of research on multiplication and its related concepts. In doing so, he seeks a conceptual analysis of the domain. In this contribution, Vergnaud clearly articulates what he means by the *domain* to be analyzed. He does not mean exclusively the mathematical concepts of "multiplication and division; linear and bilinear (and *n*-linear) functions; ratio, rate, fraction, and rational numbers; dimensional analysis; and linear mapping and linear combinations of magnitudes" (Vergnaud, p. 46), which he identifies as the mathematical bundle of topics included in MCF. He proposes to investigate these related ideas of mathematics in such a way as to include the "conceptual operations needed to progressively master this field" (p. 42), "the situations and problems that offer a sound experiential reference" (buying and sharing sweets, speed, concentration, density, similarity, probability), a "bulk" (he uses this in preference to set, which connotes too strongly well-defined borders) of concepts for analysis, and language and symbols for communicating and thinking. Vergnaud, like Steffe, locates his work in Piagetian and Vygotskian traditions and emphasizes the developmental, nonlinear acquisition of these ideas. He is explicit in his educational intentions: prediction of comparative difficulty of problems and the design of instructional situations.

Interestingly enough, both researchers end by identifying a scheme as a primary explanatory construct in their work. For Vergnaud, "Schemes are the most essential part of a theory of conceptual fields, as they generate actions" (p. 55). He

defines the role of a scheme as "the invariant organization of action for a certain class of situations" (p. 53). Schemes require "concepts in action" to provide the categories for obtaining information and "theorems in action" to allow one to derive rules and expectations.

Steffe defines a scheme, in accordance with von Glasersfeld's definition, as having three parts: "First, there is an experiential situation (i.e., a "trigger" situation, as perceived by the child, with which an activity and its result have been associated); second, there is the child's specific activity or procedure; and third, there is a result (again, as perceived or conceived by the child)" (p. 7). Steffe quotes Piaget, who wrote, "All action that is repeatable or generalized through application to new objects engenders . . . a 'scheme'" (quoted on p. 6). Steffe's goal is to "seek to learn the child's schemes and how the child might modify those schemes in the context of solving my situations" (p. 7).

Given that both researchers end up emphasizing the importance of "schemes", one might expect a measure of convergence in their claims about multiplication, division, and related concepts. However, differences are striking, perhaps a matter of degree and emphasis, but nonetheless clearly evident in the two chapters. In the Steffe work, the child is ever present. His definition of scheme emphasizes the child's perspective, and he devotes considerable attention to articulating how a child might be viewing a task as a continuation of earlier actions. Grounded on his seminal work on counting (Steffe, von Glasersfeld, Richards, Cobb, 1983), his approach starts from counting and number sequences, from which he articulates a theory of unit types to explain how a child comes to understand multiplication. Multiplication requires the coordination of two counts along with the construction of a set of objects as a unit. In the end of the chapter, he addresses the formal mathematical properties of commutativity and Krutetskii's notion of "curtailment", but he uses children's schemes to explain the properties, not the reverse.

Vergnaud's analyses are more immediately informed by the formal mathematical principles that he sees as applying across problems and representational forms as hidden mathematical structure, at different levels of abstraction. He states that the goal of teaching is to assist students in seeking and utilizing the "invariance across situa-

tions" thus promoting generalization and transfer. He explicitly rejects the idea that conceptual analysis can be conducted a priori and offers a broad outline of a developmental path through MCF. His descriptions of effective approaches come from such conceptual analysis, informed by student work and teaching situations. Vergnaud's analysis has led researchers to articulate even more differentiated maps of the field. In contrast to Steffe, however, Vergnaud does not discuss explicitly the child's voice and ways of talking. The children's methods are cast into the more formal presentation of "theorems in action" of the child.

Greer, in "Extending the Meaning of Multiplication and Division," takes the position that mathematical ideas have their birth in limited frameworks, such as the integers, and their meaning is gradually extended to allow for the inclusion of the real numbers. This claim allows him to make a fundamental epistemological claim, "that epistemological and psychological questions of comparable interest and importance are raised at every stage of reconceptualization through the long development of the conceptual field," (p. 73) and "the extension of concepts, relations, functions, and so forth from one domain to a more general domain is a characteristic mode of development within mathematics and an appropriate subject for study" (p. 74). He takes the position that categories of word problems, such as rate, multiplicative comparison, rectangular array, and product of measures, should be orthogonally considered in relation to the types of numbers used in them. Three categories are proposed: counted integers, integers or fractions derived from division, and decimals as measures of quantities.

He reviews the literature on the misconception "multiplication makes bigger and division makes smaller" (MMBDMS). Logically, he argues, the numbers in a problem that can be modeled by a single operation carry no information as to the appropriate operation. Nevertheless, they drastically influence the difficulty of the choice-of-operations task. When the multiplier is nonintegral, the problem increases in difficulty. In division, factors other than numeric values are cited as playing significant roles. Greer describes the tendency of students to alter their predictions as "nonconservation of operations" (Greer, 1987).

Again we witness a clear commitment to the situation:

"understanding of the application of the operations in modeling situations is weak" (p. 69). Greer uses the term *schema* to describe that which can be developed robustly when the student exhibits an "appreciation of the invariance of the operation over the numbers involved, which is the keystone of extension of meaning for multiplication and division" (p. 70). Citing the importance of historical or ontogenic terms, Greer takes the position that such extensions take time. He sees fractions, in particular the unit fraction, as providing the linking role, both as a divider and as a multiplier (as do Vergnaud (this volume), and Confrey (this volume) in her discussion of earlier work by Greer).

While Greer's chapter can be viewed as an elegant encapsulation of a line of studies to which he was a major contributor, in it he also leaves the reader with a set of provocative new issues to consider: (1) fractions may be useful to form a bridge to decimal multiplication and division, particularly the unit fraction, and (2) situational problems can be presented in configurations that progressively raise certain challenges to allow students to gradually overcome obstacles.

Taken together, the first three chapters of the book give the reader a variety of choices about how to conduct research in the multiplicative conceptual field. Each builds from a Piagetian framework, two using most explicitly the idea of schemes (Vergnaud, Steffe) and the third working with a conservation argument. In Steffe, the examination is built from the ground up, starting from the actions of the child and building a descriptive model informed by the discipline of mathematics, but concerned primarily with adequate explanation of the child's view of a task and methods of proceeding. In Vergnaud, an analysis of the conceptual field seeks to weave together mathematical explanation with educational purpose using the logical apparatus of "theorems in action" as the guiding construct. And Greer seeks an explanatory construct for the puzzling and robust tendency of students to alter their prediction of operation in light of the type of decimal presented. He offers "nonconservation of operation" as a psychological construct to be used in evaluating the maturity of a mathematical mind in terms of one's successful extension of understanding from the limited but intuitive class of natural numbers.

Part II. The Role of the Unit

In the second part of the book, the contributors continue a theme present in the introductory theoretical part concerning the role of the unit. The concept of a unit, an entity that is treated as a whole, is key throughout the book. Why does this particular construct become so essential in the multiplicative literature, when its importance in the addition and subtraction literature is limited? Dienes and Golding (1966), writing about "sets of sets" anticipate the need for a distinction between the entities operated on in addition and those in multiplication:

> It is important to realize that in this operation we have gone beyond the idea of addition. It is true that the same answer can be obtained to the problem by an addition of the three terms as by multiplication by three. Just because the answer is the same does not mean that the operation is the same. Multiplication involves a new kind of variable, namely the multiplier, which counts *sets*. The multiplier is a property of sets of sets. The multiplicand is a property of sets. So the two factors do not refer to the same universe. . . . Every number refers to sets in addition, whereas in multiplication some refer to sets of sets and others refer to sets. This is a very great difference and the exercise children will have had in dealing with sets and in dealing with sets of sets and even with sets of sets of sets, will serve them in good stead in coming to grips with the problems of multiplication at this stage. (p. 34)

In this volume, that distinction is strengthened through widespread discussions of the role of the unit. To create an understanding of a set of sets, one must first be able to treat a set, a collection, as a unit. This fundamental process of treating a collection as a whole is named by a variety of the authors in the volume and elsewhere as *constructing a composite unit* (Steffe, this volume), *unitizing* (Lamon, this volume), and in the context of repeated multiplication as *reunitizing* (Confrey, 1988), and *reinitializing* (Confrey, this volume). Behr, Harel, Post, and Lesh (this volume) create a way of symbolizing this action through the use of parentheses. Kaput and West (this volume) use rectangular cells in their software to represent this same action.

The emphasis in the book is that a new type of unit is

constructed in multiplication and division. If one returns to the original development of a unit in the act of a counting, a basic difference in the addition and multiplication literature is apparent. As one counts, the unit is made, and simultaneously, the unit makes counting possible. That is, the creation of number sequences is achieved by a repeated action, and from this action one creates both the numbers, 1, 2, 3, 4, 5 and the operation, a count. This interrelationship between an operation and the construction of number is an important insight expressed in a number of the chapters in the book (Steffe, Lamon, Behr et al., Kaput and West, Confrey, Smith and Confrey). In repeated multiplication, one sees a change, however. There is a repeat action, what Confrey calls *splitting*, but the initial whole is continuously revised into a new whole. In an earlier paper (Confrey, 1988), Confrey labels this *reunitizing* but later renames it *reinitializing* so as not to confuse it with the role of the multiplicative unit. This creates a recursive view of the multiplicative process in which the action of multiplication itself (with a primitive in splitting) is taken as a unit.

In the first chapter of this part, "Ratio and Proportion: Cognitive Foundations in Unitizing and Norming," Lamon recognizes the complexity of the domain of ratio and proportion and proceeds to offer two elegant constructs for organizing it. *Unitizing* is "the ability to construct a reference unit or a unit whole" (p. 92), and *norming* is "to reinterpret a situation in terms of that unit." She reports on the variety of strategies students used in solving four problems on ratio and proportion. In each case, she provides careful description of how the students approached the problems and reasoned out their answers. In her analysis and discussion, she argues that the students used a building up strategy to solve "missing value" problems, and in doing so, she suggests that they were treating the ratio itself as a unit by which they reconceptualized the problem. In her problems, in which the students are asked to compare the amounts of food distributed over a set of children and subsequently "aliens," she describes how the students used a rate (so many x per so many y) to create and build up comparisons. Her presentation of student methods demonstrates a number of her distinctions.

In the second chapter of this section, Behr et al. create

notational languages to represent the mathematics that is or might be used by children, and they demonstrate the applicability of these languages in additive and multiplicative conceptual fields. By analyzing these fields from the perspective of the unit, they point to some of their common structures. Based on this commonality they argue that using a units approach to elementary multiplicative and divisional relations (which allows for nonunitary units, i.e., composite units) would greatly enhance students' entry and successful acquisition of rational number proficiency.

They see a need for this notational language because they have found standard mathematical symbolism inadequate to represent children's mathematics in terms of the various unit types, such as those demonstrated by Lamon. Their notational language consists of two systems. The first is iconic and the second is linguistic, with the word *unit* as the basic element, from which units of units can be formed. Their use of two systems provides a stable interpretative structure and assists one in viewing the world from a "units of quantity" perspective. These systems, they emphasize, are not designed for use with children in instructional activities. They aid researchers to communicate about children's conceptions of specific additive and multiplicative situations, to hypothesize the cognitive structures that develop, or need to be developed, in acquiring an understanding of the concepts discussed, and to suggest kinds of learning events that children ought to experience so that they have an opportunity to develop these structures.

Part III. Ratio and Rate

The third part of the book concerns the topics of ratio and rate. In this part, the authors, Thompson and Kaput and West, seek to create a bridge between the role in multiplicative structures of rate, as a description across a set of particular instances, and ratio, as a multiplicative relation between two specific quantities.

Thompson's chapter on children's concepts of rate and ratio must be understood in the context of his theoretical perspective on quantitative reasoning, a perspective that brings forth the mathematics of rate, but that does not apply the ideas learned in abstract and largely symbolic

forms. In his framework, consistent with that of Steffe, Confrey, Smith and Confrey, and Kaput and West, actions are internalized into mental images of situations. The automaticity and freedom inherent in these images varies, and at the most sophisticated level, the transformation of the objects are carried out at an operational level. This progression from action to operational transformation is a key activity in Thompson's argument about the construction of quantities, which, for him, are "schematic;" that is, "a quantity . . . is composed of an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality" (p. 184). Quantification is the act of assigning a numeric value to the quality. In contrast, a quantitative operation is different from numerical operation; "it has to do with the *comprehension* of the situation" (p. 187). The comprehension of quantitative situations gives rise to the construction of quantitative operations and relationships. In the context of this theoretical framework, he deals with the question of how can one "orient a student so that she would construct a scheme for speed that would be powerful enough that she would recognize (what we take as) more general rate situations as being largely the same as situations involving speed" (p. 183).

In Thompson's report of a teaching experiment with a fifth grade girl using a computer microworld of a rabbit and a turtle, an evolution of a quantitative concept of speed is presented. Two features stand out. First, the student works toward a scheme that seems to be composed of covarying accumulations of two segmented quantities, distance and time, and no quantity acts as the primitive, or the extensive quantity in Schwartz's terminology (1988), in that, as this student built her understanding, she actually viewed time as a speed per unit distance. "[H]er initial conception of speed was that it was a *distance*, and her initial conception of time was that it was a *ratio*" (p. 224). Second, Thompson points out several important issues emerging from this teaching experiment, two of which are (1) "the standard method for introducing speed in schools as 'distance divided by time'" before students have acquired a "mature conception of speed as quantified motion," would have little, if any, relevance to their initial understanding of speed. As a result, students would

not be able to make a progress in their conception of speed as the student in his experiment did; (2) "what is called *dimensional analysis*, or the arithmetic of units (e.g., miles ÷ miles/hour = hour)" should be condemned, "at least when [it is] proposed as 'arithmetic of units,' and [we should] hope that it is banned from mathematics education" (p. 226).

In the second chapter in this part, Kaput and West report on their work with sixth grade students on missing-value proportional reasoning problems. They make a distinction similar to that made by Thompson between rate and ratio, which they term *rate-ratio* and *particular ratio*, respectively. Making this distinction allows them to describe what an understanding of rate must entail, including numerical and semantic equivalence and homogeneity of relation within the quantities being sampled.

The authors then propose a model of the development of a fuller or more mature understanding of proportional reasoning that goes from understanding particular ratios to rate-ratios through build-up strategies, abbreviated strategies, and finally the development of unit factor approaches. Going through a set of four problems, the authors articulate how a student must first identify the variables (the quantities that vary) and "The solver must then be able to form groups or segments that are the referents for the incrementing quantity steps and finally must be able to coordinate the two types of groups or segments to coordinate the dual incrementing acts" (p. 246–247).

In the two chapters, one sees similar approaches to quantity and computation, where the activity of conceptualizing the problem quantitatively precedes the computations. A second similar aspect of the two chapters is that in both approaches, one sees the beginnings of a function concept. The connections between early development of the function concept and proportional reasoning have also been articulated by Rizzuti (1991), who argued similarly to these authors, that covariation is a powerful way for students to approach contextual problems and organize a scheme to understand how quantities vary in relation to each other. This covariation approach to function resurfaces in the next chapters as the authors (Confrey and Smith) tackle the development of an understanding of exponential functions.

Part IV. Multiplicative Worlds

Confrey's chapter, "Splitting, Similarity, and Rate of Change: A New Approach to Multiplication and Exponential Functions," offers a new primitive on which the concepts of both division and multiplication can be based. Based in actions such as sharing, folding, dividing symmetrically, growing, and magnifying, the *splitting* construct is contrasted to counting actions, which create addition (affixing, joining, annexing, removing). Confrey postulates that it has a complementary but equally significant role in the elementary curriculum, one neglected under current treatment.

Her approach to multiplication shares features with others. Schemes are important as in Steffe, Vergnaud, and Thompson, but the work reported on for the early grades is largely anecdotal. She offers a cycle of conceptual construction moving from "problematic to action to reflection" describing a scheme as a "cognitive habit of action," thus stressing the repetitious quality of schemes. Explicitly, she argues for attention to operations as evolving from actions and to numbers as evolving from operations. Therefore, she sees significant interactions between situational characteristics, the actions performed by students and their purposes and goals, the development of mental operations, and subsequently the construction of numbers.

In her argument for the recognition of splitting and counting as independent cognitive structures, she notes first that in counting the origin is zero, the successor action is adding one, the unit (as the invariance between predecessor and successor) is one, addition and subtraction are basic operations, intervals are made from differences via subtraction, and rate is difference per unit time. In splitting, she suggests, the origin is one, the successor action is splitting by n, multiplication and division are basic operations, ratio is used to describe the interval between two numbers (percentages, for example), and rate is the ratio per unit time.

Like the arguments in Greer, she offers an argument that the inclusion of additional representational forms is necessary to improve students' understanding. She includes among these the tree diagram, embedded similar figures, and a new similarity based plane. She shows how her analysis can lead to a more secure foundation for the development of

the exponential function as its roots lie in repeated multiplication.

The splitting conjecture is followed by the chapter by Smith and Confrey on "Multiplicative Structures and the Development of Logarithms: What Was Lost by the Invention of Function?" Modern notation and approaches can often act as lenses that limit as well as guide our observations and investigations. In an effort to discover other approaches to multiplication, the authors examined the history of logarithms. In doing so, they discovered that the assumptions about what would be the most primitive type of comparison between two quantities or magnitudes could vary depending on the observer and the task. A simple example of this is in asking someone to compare two distances, one of which is 10 feet and the other is 13 feet. One will most frequently describe the difference additively as 3 feet more. If asked to compare the distance to the moon (238,000 miles) to the distance to the sun (93 million miles), a multiplicative comparison will typically make more sense. (The sun is nearly 400 times as far away as the moon.)

In this chapter, the authors report on "the work of Thomas of Bradwardine and Nicole Oresme, who claim that, whereas the primitive action taken on magnitudes is addition, the primitive action taken on ratios is multiplication" (p. 334). They describe a world created by Oresme where the elements are ratios and the successor action is multiplication. In doing so, they recall an earlier theme, that of functions as covariations. By juxtaposing geometric and arithmetic sequences, a logarithmic-exponential function is created. The authors point out that this is a covariation approach to function, where the functional relation is not given as input-rule-output, but as a pair located in two covarying sequences, one varying additively and one varying multiplicatively. As in the Kaput and Thompson discussions of ratios and rates, one sees how tabular approaches can support a build-up, covarying rate of change approach to functions.

Going beyond the basic placement of these two sequences opposite each other, the authors argue how density in them is created through the insertion of additive and geometric means and how Napier eventually solved the problem of allowing spacing of the geometric sequence to be of any desired interval size. The chapter is an interesting addition

to the volume for it locates the efforts of the authors in this book in a historical context.

Part V. Intuitive Models

As has been indicated in the opening of this Introduction, research on the concepts of multiplication and division revived in the early 1980s with the work of Fischbein, Deri, Nello, and Marino (1985) on the constraints that primitive intuitive model impose on students' choice of the operation to solve multiplication and division word problems. With these models, multiplication and division have very restricted meanings; for example, multiplication is conceived as merely repeated addition. Greer devotes his chapter to extensions of these meanings of multiplication and division from their early conceptualization and the difficulties children have in making these extensions. Harel, Behr, Post, and Lesh's chapter in this section report on a study of teachers' limited conception of multiplication and division. More specifically, they address the impact of the number type on the solution of multiplicative problems by preservice and inservice teachers and reexamine the findings from other studies concerning this impact, with an instrument that controls for a wide range of confounding variables. They show that teachers' solutions of multiplicative problems strongly correlate with the intuitive rules derived from the models proposed by Fischbein et al. (1985) to explain children's solutions.

Harel et al. add several points to what is known in this domain. First they looked at the explanation suggested by Fischbein et al. (1985) and others to account for differences in subjects' performance on multiplicative problems with different non-whole-number operators, and show findings that are not consistent with this explanation. They theorize an alternative conceptual basis for these differences. Second, they suggest levels of robustness of the intuitive rules derived from Fischbein et al.'s models. Finally, they raise several open research questions with regard to the impact of the number type. For example, they point out that fractions and decimals may not have the same effect on the solution of multiplication and division problems, because the naming rule of fractions is different from the naming rule of decimals: "Under these naming rules, it is easier to identify the

role of a problem quantity as an operator or operand if the quantity is a fraction than if it is a decimal; therefore it is easier to recognize its relation to other problem quantities." (p. 381).

Summary

In a final discussion chapter, Kieran articulates ways to view the overriding ideas and issues in the volume. Drawing on an exhortation from Bereiter for larger objects of conceptual analysis, Kieran writes, "The growth of multiplicative structures . . . is critical for a person's conceptualizing or bringing forth the world in which he or she lives" (p. 387). His discussion highlights the distinctions in the book between the primitives of counting and splitting and elaborates on the strengths and assumptions of each. He identifies the new emphasis in the volume on units, iteration, building-up, distributing and norming, and he suggests that the development of these new ideas might help to alleviate the difficulties identified in the research in the volume on teachers' understanding of rational numbers.

In this book, we see another step in the community's understanding of the concepts of the multiplicative conceptual field. The authors unite in their attempts to create language that will allow them to describe what children and young adults do as they encounter problems from this domain. What we see is the emergence of a significantly new language that includes such terms as *schemes, units, norming, covariation, iteration, recursion, ratio, repeated splits, similarity, sequences, operations,* and *dimensions*. These words are not commonly used in the teaching of multiplication and division, and they promise to transform such instruction in rather significant ways.

The implications go beyond the field of multiplicative relations in that an agenda is set with quantity and scheme at the forefront. Using these terms, the authors seek to assist the student in "bringing forth the world in which he or she lives" in the words of Kieran. The book suggests that close attention must be paid to how students see the tasks, not only in individual teaching experiments, but in the rich, complicated, and noisy world of classroom instruction. The classroom work reported is limited, as would be expected at the start of any reconceptualization of the territory; but the

work reported on teachers' weak understanding suggests that the need to apply this work to the classroom setting is pressing, for staff as well as students.

However, it should not be implied that the authors speak with a uniform voice. Those important differences that lead to further debate, articulation of views, and forms of evidence still are apparent in the texts. For instance, there are a variety of opinions on the relationships between quantity, operations, and numbers. At one end of the spectrum, Greer offers a description of "conservation of operation" and argues that a mathematically mature person will conserve the operation in relation to the quantities given across any number types. He suggests that this maturity is reached by overcoming the misconception that multiplication makes bigger and division makes smaller and might be facilitated by representations portraying smooth and continuous change in the magnitude of the numbers. Kaput and Thompson, in arguing for a separation of quantitative reasoning and numerical reasoning, seem to locate themselves implicitly toward the same end of the continuum, although in their empirical reports they give careful attention to cognitive jolts that occur when the segmentation of continuous quantity does not produce easy integral outcomes. These jolts, however, seem to be seen as computational disturbances and not as key in the development of the reasoning and relationships. Lamon, and Behr et al., in struggling to produce notation that can express the emergence of unit concepts, seem to be more interactional in their view of the matter. Lamon introduces new notation to help the reader avoid assuming the meaning of the division of quantity, and Behr et al. provide ample evidence of the multiple interpretations for any expression, such as 3/4. Steffe and Smith and Confrey are frankly explicit in their argument that numbers are constructed from actions and that therefore there is a necessary circularity in the number-operation relationship. The key seems to lie in one's approach to the unit. At the same time, quantities are segmented to produce units of quantity (Kaput) and units are descriptions of the invariance between a successor and a predecessor (Confrey), and both views of units are essential to the development of an understanding of multiplication and division.

The book is rich in ideas. Its implications for classroom practices are less clear. However, as a theoretical text with

careful empirical support, mostly but not completely data of the interview-based or extended teaching experiment type, the book contributes to the ongoing dialogue about how to teach these most important topics.

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