

## VARIETIES OF MATHEMATICAL DISCOURSE IN PRE-MODERN SOCIOCULTURAL CONTEXTS: MESOPOTAMIA, GREECE, AND THE LATIN MIDDLE AGES

An homage to Dirk Struik on the beginning of his tenth decade

### *Introduction*

*This first essay is an elaborated and revised version of a paper presented to a seminar, "Mathematics as a Science and as a School Subject," arranged by the Greek Mathematical Society, Branch of Thessaloniki, in April 1983. The main body of participants were Greek secondary school mathematics teachers, and, as will be seen, this audience has determined both the rhetorical surface structure and the kind of conclusions—viz., on the obligations and responsibility of mathematics education—that are drawn in Section 4.*

*The article as it stands is close to the version that was published by Science & Society in 1985. There as here, I preferred to stick to the form of the original oral exposition, letting the text present the main argument and illustrations, and hiding away documentation, conceptual clarifications, and qualifying remarks in the notes. A few points in the text have been modified, mostly for stylistic reasons or in order to avoid misunderstandings, and a few references have been added. Only in one place (note 36, the final paragraph) has genuinely new information been inserted.*

*The primary aim of the essay is to investigate how the character of mathematical thinking depends on the institutional situation in which mathematics is practiced as knowledge—perhaps as theory, perhaps as techniques one should know in order to apply them—in interplay with the wider cultural settings and societal determinants of institutions. The method is cross-culturally and cross-historically comparative, but no effort is made to find the same parameters in all cases, apart from the choice of teaching as a critical factor and institution and from the general focus on mathematics. Nor do I, indeed, believe that a schematization aiming at finding a rigid common grid of explanatory factors makes much sense in cultures as widely divergent as those dealt with here, in general character and hence also in institutional make-up. As will ap-*

pear, even the two key notions—teaching and mathematics—correspond to widely divergent phenomena, not only if we compare Mesopotamia, ancient Greece, and the Latin Middle Ages, but also when the historical development within one of these contexts is considered.

Cross-cultural comparison within limited space by necessity entails a rather coarse treatment of the single cultures. This also holds for the present article, as well as for the following one. The present essay, in addition, was written before the other items of the collection, and in some sense epitomizes the research program within which these others items belong. More fine-grained pictures will thus be found in Chapters 3 to 7.

A secondary aim of the essay is to draw explicit attention to possible consequences of the analysis that are pertinent to present social practice. Even though present concerns have been a recurrent motive for the choice of theme and approach for all articles in the collection, I have mostly left this kind of conclusion to the reader.

The dedication of the essay to Dirk Struik is due, it hardly needs to be said, to his role as a pioneer precisely in this integration of historical sociology of mathematics with contemporary political engagement and open-minded Marxist thought.

As Hellenes, you will know the Platonic view that mathematical truths exist, and that they are eternal, unchanging, and divine.

As mathematicians you will, however, also know that mathematics is not adequately described as a collection of unconnected “truths”; the whole point in the activity of the mathematician, of the mathematics teacher, and of the applied mathematician, for that matter, is the possibility to establish *connections* inside the realm of mathematics—between one theorem and another; between problems and procedures; between theorems, procedures, and sets of axioms; between one set of axioms and another set, etc.—connections that in some sense (which I am not going to discuss here) map real connections of the material or the human world.<sup>1</sup>

You will also know that such connections are established by means of proofs, demonstrations, *arguments*. Mathematics is a *reasoned discourse*.<sup>2</sup> Further, you will probably agree that the eagerness of the ancients, not least Plato and Aristotle, to distinguish scientific argument from arguments concerned with mere opinion—the arguments of the rhetor and the sophist—undermines its own purpose: What creates the need for such eager distinctions, if not the close similarity between the two sorts of arguments? On the other hand, you will also concede that *mathematical* discourse is often organized in agreement with the Aristotelian description of scientific argumentation, as argument from indisputable premises.

As teachers, finally, you will know that an argument—be it a mathematical argument—is no transcendental entity existing from before the beginnings of time. It is a human creation, building on presuppositions that in the particular historical (or pedagogical) context are taken for granted, but

which on the other hand cannot be taken over unexamined from one historical (or pedagogical) situation to another. What was a good argument in the scientific environment of Euclid was no longer so to Hilbert; and what was nothing but heuristics to Archimedes became good and sufficient reasoning in the mathematics of infinitesimals of the seventeenth and eighteenth centuries—only to be relegated again to the status of heuristics in the mid-nineteenth century (cf. Grabiner 1974).

So, one aspect of mathematics as an activity—other aspects I shall take for granted—is to be a reasoned discourse. The corresponding aspect of mathematics as an organized body of knowledge is to be the *product of communication by argument*, i.e., communication where a “sender” convinces by means of arguments a “receiver” that some statement or set of statements is true; in many cases the “receiver” is of course only a *hypothetical* average, professional interlocutor of the mathematician, defined through the sort of arguments that are thought adequate (the “model reader” of semiotic literary theory; cf. Eco 1979: 50–66). Normally, either the truths communicated, or at least some broad base for the communication, is supposed to be fixed in advance: The discourse is not fully open. In principle this sort of communication can be described as *reasoned teaching*, the concept taken as a general philosophical category. So, teaching is not only the *vehicle* by which mathematical knowledge and skill is transmitted from one generation to the next; it belongs to the *essential characteristics* of mathematics to be *constituted through teaching* in this broad philosophical sense.

However, teaching, and even teaching regarded philosophically, belongs no more in the eternal Platonic heavens than do mathematical truths and arguments. Quite the contrary. Teaching is an eminently social activity, depending on context, personal and group characteristics of the persons involved, social norms, purpose of the teaching, material, cultural and linguistic conditions and means, and the like. And so, if mathematics is constituted through being taught, we must expect it to be very much molded by the *particular* teaching through which it comes about.

It will be my aim in the following to trace that molding in the restricted sense of *institutionalized teaching*. That is, I shall eschew the airy philosophical definition of reasoned teaching where the sender convinces the receiver by means of arguments organized inside a closed or semi-closed discourse; instead, I shall concentrate on teaching as something involving a teacher and his or her students, ordered according to some fixed social and societal pattern, and normally taking place in a more or less formalized school. I shall stick to the pre-Renaissance (and so, pre-Gutenberg) era, and through three extended key episodes I shall try to trace the relations between the development of mathematics, the character of mathematical discourse, and the institutional setting of mathematical teaching (mainly the teaching of adults).

## 1. Mesopotamia: Scribal Computation, and Scribal School Mathematics

The first of my extended episodes is the development of Sumerian and Babylonian mathematics, in a process of several phases covering some 1500 years. Parallel with the rise of the early proto-Sumerian city-states in the late fourth millennium B.C., a first unification of a variety of proto-mathematical techniques (for primitive accounting, practical geometry, and measurement) into a single coherent system (*mathematics*, in our terminology), united by the common application of numeration and arithmetic, appears to have taken place. The social environment of this unification was that of the Temple corporation—indeed, the institution that molded society into a *state* was, according to all available evidence, the Temple.<sup>3</sup> It seems, however, that *mathematics* did not grow out of the mere environment of administrators of temple property and taxation: various types of evidence suggest that the unification and coherence did not really correspond to practical administrative needs, which could have been provided for by isolated extensions of the existing separate techniques. Instead, like the development of more genuine writing, mathematical unification and coherence seem to be products of the school where future officials were trained, and where the techniques that they were going to apply were also developed.

The sources for this are few and scattered, and what I have just told is a reconstruction arising from the combination of many isolated pieces of evidence.<sup>4</sup> As times go on, however, the picture becomes clearer. Toward the end of the third millennium B.C. southern Iraq had been united into a single, centralized royal state (the “neo-Sumerian Empire,” or Ur III), where the Palace directed large parts of the total economy through a vast bureaucracy. This bureaucracy was carried by a body of *scribes* who, at least since the mid-third millennium, had emerged as a specialized profession, and who had long since been taught in specialized schools. By the end of the third millennium, when sources describing the curriculum of the scribe school turn up, it is clearly dominated by applied mathematics (see Sjöberg 1976: 173). At the same time, the professional ideology of the scribal craft (as inculcated in the scribal school) becomes visible in the sources: The scribe is, or is at least expected to be, proud of his service to the royal state, which is presumed to serve general affluence and justice.

The centralized neo-Sumerian state had a short life. After only a hundred years it crumbled, not least under the weight of its own bureaucratic structure. Still, it created two very important innovations in mathematics, of which one has served since then.

The first innovation is the introduction of *systematic accounting*, on occasion with built-in controls.<sup>5</sup> Such accounting systems were created or at least very suddenly spread in the administration of the whole empire<sup>6</sup> in a

way that has rarely been equaled in history (the spread of double-entry book-keeping, an analogous process in the Renaissance, took around 300 years). The second innovation (the one that was to survive) was the introduction of the *place-value system* for both integers and fractions—not with base 10 as in our “Hindu” system, but with base 60 (as we still use it in the subdivisions of the hour and the angular degree taken over via Hellenistic astronomy). Even this system appears to have spread quickly. It was not used in official documents but for the intermediate calculations of the scribes.

Both innovations built on foundations that had been laid during centuries of scribal school activity. Regarded from that point of view, they may be said to represent mere continuity. Still, mere continuity does not guarantee theoretical progress, and so both the occasions on which progress occurs and the precise character of the progress occurring should be noticed—in this case, the ability of the scribal school to respond to the demand created by a royal administrative reform, introducing well-functioning and sophisticated tools of applied mathematics. Another fact to be noticed is the seemingly total absence of mathematical extrapolation beyond the range of applications, in striking contrast to the tendencies of the following period. The scribal school and the scribal profession appear to have been so identified with their service function to the state that no interest in mathematics *as such* grew out of a curriculum of applied mathematics, in spite of clearly demonstrated mathematical abilities. The objective requirements and regularities of mathematical structures had created a tendency toward coherence (perhaps because of the way they manifested themselves when mathematics had to be made comprehensible in teaching?); but mathematical discourse and practice seem to have remained a non-autonomous, integrated part of administrative discourse and practice until the very end of the third millennium.

This was to change during the following, so-called “Old Babylonian,” period, c. 1900 B.C. to c. 1600 B.C. As already mentioned, the centralized neo-Sumerian state crumbled under the weight of its top-heavy bureaucracy, and it was brushed aside by barbarian invasions. When organized states and civilized life had stabilized themselves once more, a new economic, social, and ideological structure had appeared. Large-scale latifundia had been replaced by small-plot agriculture, mostly held by tenants; the royal traders had become independent merchants; and the royal workshops had been replaced by private handicraft.<sup>7</sup>

This socioeconomic change was mirrored by social habits and ideology. In the neo-Sumerian Empire, only official letter-writing had existed; now, private and even personal letters appeared. The seal, once mainly a prerogative of the royal official, turned up as the private mark of the citizen. The gods, with whom one had once communicated through the temples of the royal state, now took on a supplementary role as private tutelary gods. And,

of course, the street scribe required to write personal letters on dictation appeared, together with the freelance priest performing private religious rites. All in all, the human being was no longer bound to the role of subject in the state (which he once took on when losing his roots in the primitive community); he was *also a private man*. Most strikingly, perhaps, this is seen in the case of the king. He is still the State in person, but *at the same time* he is a private person, with his private tutelary god, who may differ from the tutelary god of the State.<sup>8</sup>

Through the scribal profession and the scribal school, all this is reflected in the development of mathematics—such is at least my interpretation of the great changes in mathematical discourse and practice. The scribes remained scribes, i.e., they continued to fill their old managerial and engineering roles; the school went on to teach them the accounting, surveying, and engineering techniques needed for that; those activities and the school still imbued them with pride of serving as scribes for the royal State and the king, and so, supposedly, for general affluence and justice.<sup>9</sup> But the professional ideology of the scribes, as revealed in school texts, is not satisfied by mere usefulness. Even the scribal role has become a *private* identity. The scribe is proud of his scribal *ability* rather than of the functions that his ability permits him to carry out; and he is proud of a *virtuosity* going far beyond such abilities as would be functionally useful. A *real* scribe is not one who is able to read and write the current Akkadian tongue; these are presuppositions not worth mentioning. No, a scribe's cunning is demonstrated only when he is able to read, write, and even speak the dead Sumerian language; when he is familiar with the argot of various crafts; and when besides the normal meanings of the cuneiform signs (a task in itself, since many signs carry both one or several phonetic meanings and one or several ideographic interpretations) he knows their occult significations. All these qualities, this virtuosity, had its own name: The scribes spoke of their special *humanity*. So, a human being *par excellence*, i.e., a *real scribe*, is one who goes beyond vulgar service, one whose virtuosity defies normal understanding. He is, however, still one whose virtuosity lies within the field defined by the scribal functions; the scribe can only be proud of *scribal* virtuosity.<sup>10</sup>

You may already have thought that mathematics must be an excellent tool for anybody needing to display special abilities beyond common understanding. And indeed, the so-called "examination texts," which permit us to decipher the scribal ideology, align mathematical techniques with occult writing and craftsmen's argot.<sup>11</sup>

There is, however, more than this to the connection between the rise of private man, scribal "humanity," and mathematics. Indeed, an interest in mathematics beyond the purely useful develops in the Old Babylonian period,<sup>12</sup> leading to large-scale development of techniques at best described

in modern terms as “second- and higher-degree algebra.”<sup>13</sup> This algebra becomes a dominating feature of Old Babylonian mathematics; second-degree problems are found by the hundreds in the cuneiform texts of the period. Many of them look like real-world problems at first; but as soon as you analyze the structure of known versus unknown quantities, the complete artificiality of the problems is revealed—for example, in the case where an unfinished siege ramp is presented, of which you are supposed to know the total amount of earth required for its construction, together with the length and height of the portion already built, but not the total length and height to be attained.<sup>14</sup> These “algebraic” problems can be understood—in my opinion, can *only* be understood—as being on a par with correct Sumerian pronunciation and familiarity with occult significations of cuneiform signs: scribal ability, true enough, but transposed into a region of abstract ability with no direct practical purpose. As scribal discourse in general, mathematical discourse had been disconnected from immediate practice; it had achieved a certain autonomy.

Thus, Old Babylonian “pure mathematics” must be understood as the product of a teaching institution that no longer restricted itself to teaching privileged subjects what they needed to perform their future function as officials—i.e., it can be understood as the product of an institution where teachers and students (especially perhaps the teachers<sup>15</sup>) were also persons with a private identity as scribes.

Another aspect of Old Babylonian “pure mathematics” can, however, only be understood if we see it as a product of a *scribal* school. This aspect is the fundamental difference between Old Babylonian and Greek “pure mathematics.” Old Babylonian mathematics grew out of its *methods*, whereas Greek mathematics grew out of *problems*, to state things very briefly.

This may sound strange. Indeed, Babylonian mathematics is known only from problems; it does not contain a single theorem and hardly a description of a method.<sup>16</sup> That, however, follows from the training role of the Babylonian school, which did not aim at the theoretical understanding of methods but at the *training of methods*—first, of course, the training of methods to be used in practice, but next also of methods that would permit the solution of useless second-degree problems. Such training could only be obtained through drill. Indeed, the problems that occur appear to be meant exactly for drill; at least, many of them appear to have been chosen not because of any inherent interest but just because they *could be solved* by the methods at hand.<sup>17</sup>

Greek mathematics, on the other hand, for all the theorems it contains, grew out of problems for the solution of which methods often had to be created anew. We may think of the squaring of the circle, the trisection of the angle, and the Delic problem (doubling the cube). But we should not forget

the Eudoxean theory of proportions or Book X of the *Elements*, on the classification of irrationals and the algebraic relations between the resulting classes.<sup>18</sup> They, too, are investigations of problems—problems arising from and only given meaning through the development of mathematical theory.

The Greek effort to solve fundamental problems is clearly related to the whole effort of Greek philosophy to create *theoretical understanding*. That could never be the aim of a scribal school. There, skillful handling of methods was the central end. Problems were the necessary means to that end, and in the Old Babylonian school they became the necessary pretext for the display of virtuosity.<sup>19</sup> Paganini, not Mozart, plays the violin.

*Mutatis mutandis*, we can speak of a Babylonian parallel to the effects of the “publish-or-perish” pressure on contemporary publication patterns. When submitted to such pressure, the easiest way out is to choose your problems according to their accessibility, given your ability and the methods with which you are familiar; current complaints that the scientific literature is drowned in publications treating problems of no other merit than that of treatability indicate, even if great exaggeration is allowed for, that some researchers have found this way out.<sup>20</sup>

Apart from its determination by methods rather than problems, another characteristic of Old Babylonian “pure mathematics” seems to derive from its particular background: *viz.*, that you have to analyze the structure of its problems in order to decide whether they are practical or artificial, *i.e.*, “pure” (cf. the siege-ramp problem mentioned previously; superficially regarded, it looks like a typical piece of militarily applied mathematics). In other words, even when Old Babylonian mathematics is “pure” *in substance*, it remains *applied in form*. In contradistinction to this, the prototype of Greek mathematics is pure in form as well as in substance, to such an extent that the applications of geometry to astronomy are formulated abstractly as dealing with spheres in general—so in Autolykos’s *Moving Sphere* and Theodosios’s *Spherics*.<sup>21</sup> Greek mathematics had become pure in form, *i.e.*, fully abstract, even when it was applied in substance.

This “applied form” of Old Babylonian mathematics could hardly have been different; at least, it is in full harmony with other aspects of scribal “humanism.” The virtuosity to which a scribe could aspire had to look like *scribal virtuosity*. The scribe, however, was an *applier of mathematics*, a *calculator*; there were no social sources, and no earlier traditions, from which a concept of mathematics as a concept *per se* could spring, and there was thus no possibility that a scribe could come to think of himself as a *virtuoso mathematician*. Only the option to become a virtuoso calculator was open; so, Babylonian “pure mathematics” was in fact calculation pursued as *art pour l’art*, mathematics applied in its form but disengaged from real application. Expressed in other terms, Old Babylonian mathematical discourse had



achieved autonomy for its actual working; it remained, however, *defined* through scribal professional practice.

So much for Sumerian and Babylonian mathematics. The argument could be supported by telling the story of Babylonian mathematics after the disappearance of the individualized Old Babylonian social structure and of the scribal school as an institution—"pure" mathematics vanishes from the sources for more than a thousand years—or by comparing Egyptian and Babylonian developments. This, however, I shall bypass,<sup>22</sup> and I shall close my first episode by emphasizing that the overall social characteristics of the institution in which Mesopotamian mathematics was taught and developed influenced primarily the overall formal characteristics of Mesopotamian mathematics as a discourse and as a developing system, and normally *not* its factual contents. The Babylonian and the Greek would calculate the diagonal of a rectangle with sides 5 and 12 in the same numerical steps; so far, of course, mathematics *does* consist of socially and historically transcendental truths. But whereas the Greek mathematician would argue from a strict concept of the rectangle built upon a concept of quantifiable angles, the Old Babylonian scribe would only distinguish quadrangles with "right" angles from those with "wrong" angles.<sup>23</sup>

## 2. Greek Mathematics: From Open, Reasoned Discourse to Closed Axiomatics

My next episode will be the development of Greek mathematics. Whereas the role of argumentation in the Mesopotamian development follows mainly from indirect evidence supplemented by a few clay tablets where something like a didactical explanation occurs,<sup>24</sup> Greek mathematics is argument through and through.

Greek mathematics is thus a product of "reasoned teaching" in the general philosophical sense that I gave to this expression. It is more dubious whether it is a product of institutionalized teaching; it may be that the overwhelmingly argumentative character of Greek mathematics should be sought in its initial *lack* of school institutionalization.

At first, however, a very different hypothesis suggests itself. Much of Greek mathematics has been seen as a search for harmony and completeness—and do these ideals not look as if they were taken over from the *paideía eleúthera*, the liberal education of the free citizen as a harmonious and complete human being?<sup>25</sup> Furthermore, once institutionalized (from the fourth century B.C. onward) the *paideía* came to contain a fair measure of mathematics. Finally, Proclus tells us that Pythagoras gave mathematics the "character of a liberal education," *schêma paideías eleuthérou*.<sup>26</sup>

On closer investigation, however, there seems to be no causal connection leading from the *paideía* to mathematics. Apparently, we are confronted with two analogous but distinct structures, partly with a causal chain leading the opposite way, partly perhaps with a neohumanist optical illusion. Therefore, instead of analogies, I shall try to build my exposition on the actual evidence,<sup>27</sup> incomplete as it is.

I shall try to distinguish three periods, of which those of most critical interest for the investigation are unhappily the most difficult to document:

- The rise of reasoned mathematics, in the sixth and fifth centuries B.C.—“pre-Socratic mathematics.”
- The creation of deductive and axiomatic mathematics, in the fourth and early third century—from Plato and Eudoxos to Euclid.
- Finally, the mature period from Euclid onward, in which the style and character of Greek mathematics was already fixed, and in which every Greek mathematical text known to us, a few fragments aside, was created.<sup>28</sup>

The first period is that of the Pythagorean order, the philosophical schools, and the sophists. Of these, the sophists, who were creators of a theoretical concept of education, and whose *paideía* aimed exactly at the creation of the complete human being or citizen, have the least to do with mathematics. Truly, Hippias, Bryson, and Antiphon interested themselves in the trisection and circle-squaring problems. But Hippias's curve for trisection<sup>29</sup> is nothing but a smart trick, and Bryson's and Antiphon's treatments of the squaring shows them (according to Aristotle's polemics) to lie outside the mainstream of Greek mathematical thought.<sup>30</sup> One is tempted to assume that the sophists' treatment of these problems reflects the necessity for those professional teachers to deal with mathematical subjects *à la mode* in order to satisfy their clientele. If such is the case, sophist education underwent the influence of mathematics, not the reverse.

In the case of the Pythagorean order,<sup>31</sup> it is difficult to distinguish legend from history. It seems sure, however, that the Pythagorean movement was the place where *mathémata* changed its meaning from “doctrines,” i.e., matters being learned, to “knowledge of number and magnitude,” i.e., to “mathematics,”<sup>32</sup> no later than the late fifth century B.C. It is well established that an essential part of the teaching of the order, be it secret or not, was by then dedicated to theoretical arithmetic, to geometry, to harmonics, and to astronomy—the four *mathémata* listed by Archytas. Finally, the active role of the Pythagoreans in the development of arithmetic, harmonics, and astronomical speculation during the fifth century can be trusted with confidence.<sup>33</sup>

According to tradition,<sup>34</sup> the order consisted of an inner and an outer circle, *mathematikoí* and *akousmatikoí*, “mathematicians” and “listen-

ers"—if the two groups were not, as seems more likely, the result of a split in the order. The latter are supposed to have been literal followers of the tradition, whereas the former are supposed to have been taught rationally, and perhaps to have made rational inquiry (a supposition which is independent of the question whether their group went back to Pythagoras himself, or was a later fabrication). If this tradition is reliable—as it appears to be—the teachings of the *akousmatikoi* will at most have contained semi-mystical numerology, more or less shared with folk traditions. Some Pythagoreans may, on the other hand, have started rational inquiry from this basis; and their investigations of theoretical arithmetic, of harmonics, and of the problems of the irrational, may have become part of a cumulative research tradition *because of* integration with a stable tradition for reasoned teaching of the *mathematikoi* (the tradition giving rise to the very name of the group). More than this can hardly be said, given the lack of adequate sources.<sup>35</sup>

Still, I find it doubtful whether the rational and abstract character of Greek mathematics can have originated inside the circle of *mathematikoi*. It would seem to be in better harmony with the picture of the Pythagorean order as a mystical, religious, and ethico-political movement, if available philosophical knowledge, including mathematics, was borrowed from outside in cases where it could serve the overall world view and aim of the movement (cf., for example, Schuhl 1949: 242–57)—possibly in a process of several steps, where the most elementary abstract arithmetic was taken over (as numerology and for use in musical and cosmological speculation) already in the late sixth or early fifth century. In fact, various sorts of evidence points in that direction.<sup>36</sup> It thus appears that the mathematical activity of the Pythagoreans consisted of work in agreement with the rational tradition when it was already established, and refinements of the same tradition. Therefore, I shall direct attention to the open philosophical “schools.”

Here, a word of caution may be appropriate. The philosophical schools were probably not schools in an institutional sense. Even when we go to the mathematical “schools” of the fourth century, what translators designate, e.g., “the school of Menaechmos,” is spoken of by Proclus as “the mathematicians around Menaechmos.”<sup>37</sup> No doubt the philosophers had disciples; but they are distinguished from the sophists by not being *determined* as professional teachers.<sup>38</sup> The philosophers made rational inquiry, some of them in mathematics; and they taught. Both activities must be understood as implying rational discourse. But nothing indicates that the philosophers’ inquiry was determined in style and structure by their teaching.

Instead, the reasoned and abstract structure of fifth-century mathematics and its orientation toward *principles* must be sought elsewhere. That such a tendency was there is obvious even from the scant source material at our disposal, be it Hippocrates’ investigation of the lunes<sup>39</sup>; his writing of the presumed first set of *Elements* (Proclus, *In primum Euclidis* 66<sup>7-8</sup>);

Oenopides' presumed first theoretically founded *construction* of dropping and ascending perpendiculars and his singling out of ruler and compass in that connection<sup>40</sup>; the description of the solar movement in the ecliptic as an inclined great circle (equally ascribed to Oenopides<sup>41</sup>); or a number of Platonic passages, from the references to investigations of incommensurability (*Theaetetus* 147d–148b; *Laws* 819d–820c) to the slave boy guided by suggestive questions to the doubling of a square (*Meno* 82b–85b).

It is also evident that all these pieces of evidence point toward various locations inside the "philosophical movement." The immediate background to the rise of reasoned mathematics is thus the condition of mathematical discourse and inquiry as part of general philosophical discourse.

This observation is also in harmony with chronology. Even though a number of mathematical discoveries are ascribed to Thales, and such ascriptions can neither be proved nor disproved (not least because it is not clear what, precisely, Thales is supposed to have discovered), the philosophical transformation of myth and cosmogony into philosophical cosmology seems to precede the rise of genuine reasoned mathematics. Indeed, even the Eleatic critique of natural philosophy appears to precede, if not the first steps toward a reasoned approach to arithmetic, then at least the techniques of proof that came to characterize Greek mathematics as we know it (cf. note 36).

So, the rationality of fifth-century Greek mathematics appears to build on general philosophical rationality, and on an open, non-hierarchical type of discourse: Not the one-way master-student relationship of institutionalized teaching, but a discourse of mutual disagreement, conflict, and common search.<sup>42</sup> Whereas the former may be more effective for the assimilative expansion of a knowledge system, the latter, open discourse, may be the presupposition for fundamental change.<sup>43</sup> So, if I am right in my interpretations, the *specific* formation of Greek mathematics may have originated in the *lack* of didactical institutionalization of the soil from which it grew during the first phase, as claimed already.

The open-type discourse of the philosophical environment may have had wider, social backgrounds: perhaps not so much in fixed social institutions as in the break-up of institutions. Indeed, the Solon reforms, which averted social conflict by instituting reasoned constitutional change, are contemporary with the earliest Milesian philosophy—and they are not the first attempt of their kind (cf. G. Smith 1956). About a century before Solon and Thales, Hesiod presents us with an instance of conceptual analysis by dichotomy,<sup>44</sup> in a way that reminds one very much of Plato's dialogues, but which in historical context shows that the germs of logical analysis are older than philosophy in Greece. Still earlier, at the dawn of Greek literature, the rhetoric of the Homeric heroes contains clear dialectical, syllogistic figures (used even to persuade the gods).<sup>45</sup> Ultimately, the discourse of early Greek

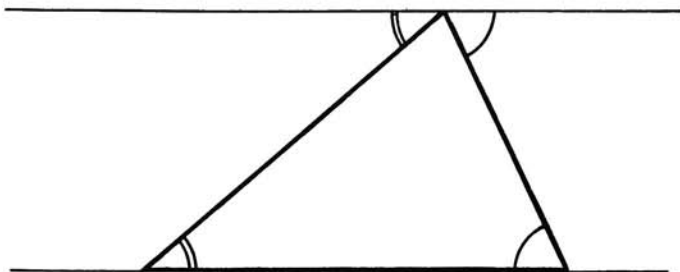
rational philosophy may go back to the open discussion of the popular assembly and the *agora*<sup>46</sup>—whereas, as we saw, the discourse of Mesopotamian mathematics, *explaining* procedures and *training* scribes rather than *investigating problems* or questioning, goes back to the more closed discourse of organized school teaching.<sup>47</sup>

The second phase of the development of Greek mathematics, going from Archytas, Plato, and Theaetetus, not only brings a marked quantitative growth of mathematical knowledge, explained by Proclus/Eudemos as an increase in the number of theorems (*In primum Euclidis*, 66<sup>16f</sup>), but also a fundamental qualitative change, “a more scientific arrangement.”<sup>48</sup> Indeed, what happens is a continuation and accentuation of a process inaugurated by Hippocrates of Chios when he wrote the presumably first set of *Elements*. Gradually, mathematics comes to consist of larger, theoretically coherent structures; no longer just reasoned, it becomes deductive and, in the end, axiomatic. The ideals for the organization of mathematical knowledge are clearly delineated by Aristotle in the *Analytica posteriora*, where these ideals are even put forward as paradigmatic for all “scientific” knowledge (cf. note 48).

No doubt the search for coherent structures gave extra impetus to the quantitative growth of knowledge; and probably the quantitative growth called for better organization. But this internal dynamic of late fifth and early fourth century mathematics was only made possible because mathematics had become something possessing a *social identity*. Mathematical activity had become institutionalized; the very successes of mathematics toward the late fifth century had made it a field of learning of its own.

Mathematics was institutionalized on several levels. It was introduced into the *paideía* of adolescents; but that was in all probability without effect on the dynamics of mathematical knowledge (apart from the recruitment thereby procured), the mathematics taught to adolescents being quite elementary. But mathematics also became something that one might study as a philosopher taught by a teacher—e.g., as one of the “mathematicians around Menaechmos.” The increasing systematization of reasoned and argumentative mathematics performed as an autonomous activity automatically led toward deductivity and axiomatization. Systematization relentlessly revealed flaws and circularities in argumentation; no mathematician gathering a circle around himself could then avoid trying to get rid of such defects.

In the phase of merely reasoned mathematics, it would have been possible to prove that the sum of the angles in a triangle equals two right angles by drawing the parallel without questioning its existence<sup>49</sup>; in other connections, it would be possible to argue for the existence of a parallel in a way that ultimately involved knowledge of the sum of the angles of a triangle. In fact,



such attempts to prove everything are discussed by Aristotle in *Analytica posteriora* (72<sup>b</sup>33–73<sup>a</sup>20). When geometry became an integrated system, the circularities arising from the combination of such piecemeal demonstrations would become evident. Even this is borne out by Aristotle when he speaks of “those persons who do think that they are drawing parallel lines; for they do not realize that they are making assumptions which cannot be proved unless parallel lines exist” (*Analytica priora* 65<sup>a</sup>5–7). The recognition would then force itself upon the mathematicians that some things had to be presupposed, in agreement with the initial sentence of the *Analytica posteriora*, that “all teaching and learning that involves the use of reason proceeds from pre-existent knowledge” (71<sup>a</sup>1–2). In the end, Euclid found a way out of the complex problem by a combination of his fifth postulate (which implies that at most one parallel exists) and a tacit assumption about the figure used to prove prop. I.16—an assumption that holds true only in geometries where parallels exist (and hence not on a sphere).

Trying to describe the character of mathematical discourse of this second phase, we can say that it becomes closed into itself, i.e., autonomous: Mathematics builds up its own system of scientific and epistemological norms; when Protagoras argues against the geometers that “the circle touches the ruler not at a point [but along a stretch of finite length],”<sup>50</sup> he just disqualifies himself (in the eyes of fourth-century geometers) by being unaware of this closing of mathematical discourse—dealing, as Aristotle argues, not with sensible and perishable lines (etc.) but with the ideal *line in itself*—to non-mathematical reason.<sup>51</sup> In the elementary *paideia*, the mathematical discourse also becomes closed in the sense of being one-way, dependent on authority and open to no alternative thinking. The same process is on its way at the philosophical level, but here only as a goal pursued: Mathematics is understood as concerned with eternal, immovable truth, and thus it cannot admit of alternatives and discussion of its foundations; so, mathematics must by necessity *aim* at the closure of its own discourse.<sup>52</sup>

By the end of our second period, this process was carried to its end. Already in the outgoing fourth century, “a common fund of theorems existed, in plane as well as spherical geometry, which had already taken on their

definitive form, and the formulation of which was to be perpetuated for centuries with no change whatever" (Aujac 1984: 10). In the early third century B.C. (or, if recent proposals to make Euclid a contemporary of Archimedes are correct, around 250 B.C.—see Schneider 1979: 61f.), Greek mathematical thought has been shaped in the Euclidean *Elements* as a specialized, hard, sharp, and immensely effective tool for the production of new knowledge. In the third period, it produced astonishing quantitative accretions, e.g., in the works of Archimedes and Apollonios. But its whole style and formal character was fixed. It was deductive, axiomatic, abstract, and formally "pure"; and it was totally "Euclidized." Commentators such as Pappos, Proclus, and Theon of Alexandria explain and extend (for good or for bad); *grosso modo* they raise no doubts. Truly, Archimedes extended the range of aims of mathematics by his numerical measurement of the circle, in a way that (in spite of its "pure" form) was noticed as a deviation by some commentators (see Vogel 1936: 362). But even if his results were adopted, they inspired no further renewal (apart from the Heronian tradition, the connections of which to Archimedes are clear, but which misses on the other hand the high level of mainstream Greek mathematics). Greek mathematics had, in the *Elements*, got a paradigm in Thomas Kuhn's original sense of that word: a book "that all practitioners of a given field knew intimately and admired, achievements on which they modeled their own research and which provided them with a measure of their own accomplishment" (1963: 352). Mathematical discourse now became really closed, in agreement with the "Platonic" intentions; the closing was, however, not effected by a teaching institution, but through a book—or, better, it was effected by a teaching institution whose main institutional aspect was the use of that book.

This institutionalization did not give Greek mathematics of the mature phase any *new* character. But it was the precondition for the perpetuation of a character that Greek mathematics had once acquired through a series of settings which had now disappeared: institutionalized and non-institutionalized, discursively open and discursively closed.

The social carriers of mathematical development in the mature phase had less to do with teaching, institutionalized or not, than their predecessors of any earlier phase. Only a few, and not the greatest, were connected to those institutions of higher learning that, from Plato's Academy onward, had succeeded the earlier philosophical circles or informal schools. Most great mathematicians are best described as "professional full- or part-time amateurs" (strange as this mixing may sound to modern sociological ears), who after a juvenile stay, for example, at the centers of higher learning in Alexandria remained in mutual contact through letters, and whose impregnation with the professional ideals of the discipline had been so strong that their lonely work and their letters could maintain them as members of one stable scientific community.

### 3. The Latin Middle Ages: A Discourse of Relics

In order to prevent any simplistic—or just simple—picture from emerging, I shall discuss one more episode. I shall bypass the very interesting relations between socioeconomic and cultural background, traditions, and institutions, types of mathematical discourse, and the development of mathematics in ancient and medieval India and in the medieval Islamic world.<sup>53</sup> Instead, I shall concentrate on the Latin Middle Ages of Western Europe.<sup>54</sup>

The Roman part of the ancient world had never shared the Greek interest in theoretical mathematics; as Cicero remarks, the Romans restricted their mathematical interests to surveying and computation.<sup>55</sup> Truly, the education of a Roman gentleman was built on the seven Liberal Arts of the Greek *paideía*: grammar, rhetoric, dialectics (i.e. logic), arithmetic, geometry, harmonics ('music'), and astronomy. But what was taught in the latter four mathematical arts was utterly restricted. Only a few of the Greek mathematical works were ever translated into Latin (only one of which, part of the *Elements* translated by Boethius around A.D. 500, was of scientific merit), and only superficial popularizations were ever written in Latin.

The Christian takeover of education in late antiquity did nothing to repair this: on the contrary. Still more unquestionably than the gentleman, a good Christian should definitely not be secularly learned, even though good manners required him to be culturally polished. The breakdown of the Western Empire and the rise of loose barbarian states deprived him even of the possibility to be polished.

Still, if not saving much of the ancient learned legacy, the Christian Church saved at least the faint memory that *something had been lost*. At every occasion of a cultural revival, be it Visigothic Spain in the early seventh century, Anglo-Saxon England in the early eighth, the Frankish Kingdom of the early ninth, or the Ottonian empire of the late tenth, the recurrent characteristic is an attempt to reconquer the cultural ground that had been lost. This is the reason why every revival looks like a renaissance and has been labeled so by its modern historians.

Until the end of the first millennium, the reconquests at the mathematical front were restricted to arithmetical Easter calculation (*computus*); ancient presentations of theoretical arithmetic and harmonics for non-mathematicians; and some surveying manuals that had to play the role of geometry. Now, by the onset of the High Middle Ages, things were going to change. But in the actual moment, mathematics was (like every remnant of ancient culture) as much a sacred relic as something to be learned and understood or as a type of discourse; furthermore, even as a subject to be learned or as a discourse, mathematics was profoundly marked by being a relic in a culture given to worshipping relics.<sup>56</sup>



An economic and demographic leap forward in the eleventh and twelfth centuries was the occasion for a revival of trade and monetary economy and for the rise of towns achieving a certain degree of autonomy (ideological autonomy, often officially recognized juridical autonomy, and, in economically advanced regions, even *de facto* political autonomy). Men participated in the social and political life of these towns as members of more or less institutionalized groups, inside which they acted as equals; presumably it was on this background that an interest in open reasoned discourse grew in the eleventh century urban environment.<sup>57</sup>

The economic revival was also the occasion for the growth of *cathedral schools*, the students of which were taught the seven Liberal Arts to the extent that competent teachers and the necessary text materials were at hand. From the point of view of the Church, the schools were designed to provide future priests and other ecclesiastical functionaries with the knowledge necessary in a new social context where the priest had to be more than the main actor of rituals and the magician of relics.<sup>58</sup> At the same time, the growth of the schools can be seen as yet another spontaneous expression of the recurrent tendency to translate cultural blossomings into ‘renaissances,’ revitalizations of ancient learning. Finally, the clerks trained at the cathedral schools would often come to serve in and outside the Church in cancellarian and secretarial functions, the non-engineering aspect of the old scribal function. As long, however, as the ideological interest in free discourse and the cultural need for a renaissance prevailed, the schools did not take on the character of scribal training schools. The disciplines of autonomous thought, which had not been known to the Babylonians, were now at hand, and they were fundamental to the scholars’ view of their world and of their own identity.

For mathematics, the eleventh century school meant little directly, apart from a firmer possession of the insecure conquests of the late first millennium. An awakening interest in astrology, nurtured by a few translations of Arabic treatises on the subject in the tenth and eleventh centuries (see J. W. Thompson 1929 and van der Vyver 1936), is probably best understood as an expression of the search for natural explanation distancing direct divine intervention,<sup>59</sup> an essential search in a society on the way to rationalize its world picture. Even though partly carried by cathedral school masters, it was no product of the school institution as such, whose whole heritage both from ancient learning and from the Fathers of the Church would rather have made it separate liberal-arts astronomy from astrology.<sup>60</sup>

Indirectly, though, the eleventh century school had great importance for the future of mathematics. Taken together, the rational, discursive spirit of the times and the training and opportunities provided by the schools gave rise to great changes in every corner of Latin learning. On mainly native ground, figures such as Anselm of Canterbury, Abelard, Hugh of Saint-

Victor, the "twelfth century Platonists," and Gratian recast philosophy, theology, and canon law in the late eleventh through the mid-twelfth century. More important for mathematics, the background provided by the schools made possible the translation of Muslim learning and of Arabic versions of ancient Greek works (and, initially to a lesser extent, direct translations of Greek works) by creating the scholarly competence and, not least, the enormous enthusiasm of the translators; furthermore, the schools provided a public that could receive the translations and have them spread. So, during the twelfth century, most main works of ancient and Judeo-Muslim astrology (including Ptolemy's *Almagest*), the *Elements*, al-Khwārizmī's *Algebra*, and several expositions of "Hindu reckoning" (the decimal place value system for integers and its algorithms) were, together with many other mathematical and non-mathematical works, translated and spread.<sup>61</sup> Even the larger part of Aristotle's *Organon*, his *Metaphysica*, and part of his natural philosophy were first transplanted in the mid- to late twelfth century.

In the late twelfth century, learning at those cathedral schools which were to develop into universities was in a situation of suspense. Learning was not seen as something being created in an active process. Learning already existed in the form of great scholarly works, "authorities;"<sup>62</sup> the enthusiasm for knowledge still had something of the enthusiasm for relics.<sup>63</sup> And yet, the "new learning" was really something *new*; as a body of relatively *coherent* knowledge it was in fact something that was being actively created. In spite of its concentration on already existing authority, the discourse of the new learning was anything but closed: There were still tight connections between the open "political" discourse of the corporations ("universities") of masters and students and the discourse of learning.<sup>64</sup> The "political" discourse of the scholars, for its part, was of the same genre as that which had manifested itself in the turbulent urban environment already in the later eleventh century, reinforced in a synergetic process by the increasingly "dialectical" organization of the teaching institution.

The openness and the semi-political character of the learned discourse of the late twelfth and the early thirteenth century was not mistaken at the time. As Bernard of Clairvaux had attacked the rationalizing theology of Abelard and the "Platonist" philosophers, so many smaller theological minds attacked the new learning, complaining, for example, that the Christians (and even monks and canons) not only wasted their time but also endangered their salvation studying the

philosophical opinions, the [grammatical] rules of Priscian, the Laws of Justinian, the doctrine of Galen, the speeches of the rhetors, the ambiguities of Aristotle, the theorems of Euclid, and the conjectures of Ptolemy. Indeed, the so-called Liberal Arts are valuable for sharpening

the genius and for understanding the Scriptures; but together with the Philosopher they are to be saluted only from the doorstep. [ . . . ] Therefore, the reading of the letters of the Pagans does not illuminate the mind, but obscures it.<sup>65</sup>

It will be observed that even interest in the *Elements* is presented as a danger to theological order (and, beyond that, to social order in the ecclesiastical universe).

That, however, was only for a brief period. Euclidean mathematics was not fit to serve the construction of a coherent counterdiscourse to the discourse of the conservative theologians. Interest in mathematics dropped back from the front in thirteenth century learning. The great conflicts, the prohibition of dangerous works, and the executions of heretical scholars were all concerned with Aristotelian philosophy and its derivations (including pseudo-Aristotelian occult science).<sup>66</sup> Except for a few active researchers, the scholars of the thirteenth century looked at mathematics as a venerated part of the cultural heritage. In the context of the thirteenth century university world, mathematics was better fit as a modest part of the peaceful synthesis than as a provider of revolutionary counterdiscourse.

Such synthesis did take place in the mid-thirteenth century. It took place at the social and political level, where the liberation movement of the towns attained a state of equilibrium with the prevailing princely power, and where even princely, papal, and feudal power learned to live together; and it was seen at the level of religious organization, where the specific urban spirituality got an authorized expression through the orders of friars, but where it lost its *autonomous* expression through lay pauper movements. Closer to our subject, synthesis forced its way through in the matter of Aristotelian philosophy. By the 1230s, the position of the conservative theologians had become untenable: Aristotelian metaphysics had penetrated their own argumentation to the bones. But for all the fluidity and turbulence of the university environment, general social and ideological conditions were not ripe to overthrow the power of the ecclesiastical institution. This scholarly stalemate was solved through the great Christian-Aristotelian synthesis due to Saint Thomas and Albert the Great. Thanks to their work, the "new learning" was fitted into the world conception of Latin Christianity, as that cornerstone which had been missing (and missed) for so long. One is tempted to paraphrase St. Thomas's famous dictum, *Gratia non tollit naturam sed perficit*, to the effect that "natural philosophy did not abolish the world view of Divine Grace; it made it complete."

By making it complete, however, Latin Aristotelianism was given an orthodox interpretation that deprived it of its character of open discourse. Only for a short time (roughly speaking, the fourteenth century) was it able

to develop answers to new questions, and to procure a world-view that could really pretend to being a view of a world in change; after that, Aristotelian learning (and the whole of medieval university learning, which had come under its sway) was upheld only by institutional inertia and by the external social forces guaranteeing its survival. Gradually, the universities were to develop into training schools for priests, lawyers, physicians, and officials, of scribal-school character. Their much-beloved dialectical method, once the reflection of an open, critical discourse, could be derided as a display of empty virtuosity by the satirical authors of later ages, from Thomas More to the eighteenth century.

Mathematics was no main pillar in the synthesis. But on a modest level, it had prepared the way. Neither the mathematics of the "new learning," nor mathematics at all or its single constituent disciplines, were ever practiced on a larger scale as something autonomous. The traditional mathematical disciplines belonged to the total scheme of Liberal Arts. The commentaries written into mathematical treatises, be it the eleventh-century low-level presentations of theoretical arithmetic (see Evans 1978) or the twelfth- and thirteenth-century *Elements* (see, for instance, Murdoch 1968), demonstrate their attachment to a teaching tradition where the establishment of connections to the non-mathematical disciplines of the curriculum were as important as mathematics itself. These traditional disciplines remained—the short-lived tendencies of the twelfth century and a few mathematicians by inclination<sup>67</sup> disregarded—integrated parts of a larger cultural whole, and parts of a heritage. Mathematical discourse was never, as it had been in antiquity from the fourth century B.C. onward, autonomously closed on itself; its main epistemological responsibility was not inwardly but outwardly directed.<sup>68</sup> But it was closed on past performances, closed to fundamental renewal, closed to alternatives.

The non-traditional disciplines were necessarily less closed: Algebra, optics, the "science of weights" (i.e., mathematical statics) did not fit into the traditional curriculum and could only be merged with it through a creative process. But only a handful of people engaged in these disciplines, and of the works of that handful of scholarly eccentrics, only those that were congruous with traditional thought gained general acceptance.<sup>69</sup>

The prevailing scholarly synthesis was a real synthesis of all the important interests present: Only a few of those concerned did not feel good inside its frame. Those who did not (and their number increased as the fourteenth century approached and especially during its progress) would rather leave the universe of closed rational discourse altogether and fall into mysticism and skepticism, than try to open it to alternative rationalities—be they mathematical or philosophical.